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W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

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AND

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## CONTENTS.

	PAGE
AEROPLANE MATHEMATICS. By S. BRODSTEIN, PH.D., M.A.	257
REVIEW. By Lt.-Col. R. K. HEELER; E. M. LANGLEY, M.A.; G. B. MATHEWS, F.R.S.; PROF. E. H. NEVILLE, M.A.	282
THE LIBRARY,	288

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AEROPLANE MATHEMATICS.

1. The dynamics of aeroplane motion is a comparatively new subject, and one that arouses interest in all students of mechanics. Although a full study of aeronautics involves the use of complicated analysis, yet some really useful information about aeroplane motion can be obtained with quite simple mathematics. It is the object of this paper to draw attention to such methods and results. It will be seen that some of the investigations can be understood by mere beginners in mechanics; others are more suitable for such students as have had considerable experience of mechanics.

Everybody is familiar with the general shape of an aeroplane. When the aeroplane is flying in a straight line we recognise in it a plane of symmetry,

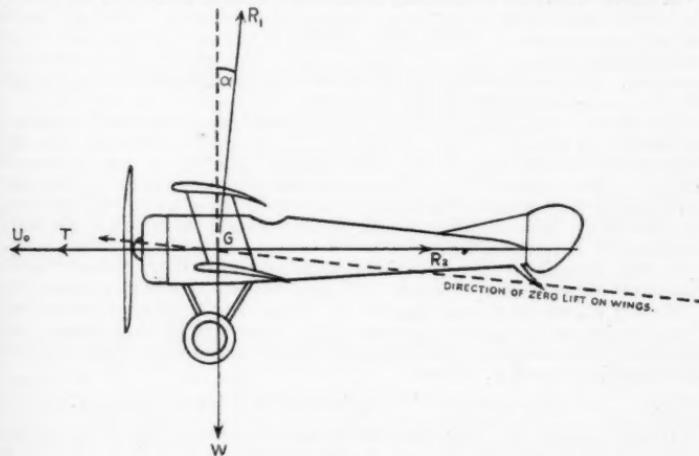


FIG. 1.

which is then vertical. We call this the *longitudinal plane*, a term which will remain applicable when the symmetry is destroyed, as in circling and spiral flight (§§ 14-19). Fig. 1 shows a biplane seen in elevation on the longitudinal

plane. We take the machine to consist of the *wings*, the *body* (including under-carriage, etc.), and the *tail*. We take the tail to consist of an *elevator*, which turns about an axis perpendicular to the longitudinal plane, and a *rudder*, which turns about an axis in the longitudinal plane. In practice part of the horizontal tail is fixed and part of the vertical tail is fixed ; we include these in the body.

There is a well-defined line in the longitudinal plane : this is the *axis of the propeller* or air-screw. We assume that this axis passes through the C.G. of the machine, as well as through the line about which the elevator turns. When the elevator is so situated that the propeller axis lies in its plane, then we have the *neutral* position of the elevator.

As regards air-pressure we make the following assumptions. If the elevator is neutral and the motion of the machine through the air is along the propeller axis, then there is no pressure on the elevator. If the rudder is in the longitudinal plane and the motion is also in this plane, there is no pressure on the rudder. The air resistance due to the body of the machine is taken to act along the propeller axis, backwards, *so long as the motion through the air is not too much inclined to this axis* : this resistance is proportional to the square of the speed through the air.

The air-pressure on the wings needs more careful definition. There is a certain direction of motion in the longitudinal plane for which there is no lifting effect on the wings, which means that the air-pressure has no component perpendicular to this direction. If the actual direction of motion in the longitudinal plane makes an angle with this direction, we call this angle the *angle of attack*. We assume that the air-pressure on the wings is proportional to the density of the air, the square of the speed, and the *sine of the angle of attack*. We also assume that this pressure is *normal* to the direction of no lift, and that the line of action always passes through the C.G. of the machine. We have here imported considerable simplification ; but it will be seen that many of the more important features of aeroplane motion will emerge nevertheless.

**2. Steady Flight for Symmetrical Aeroplane ; Normal Flight.**—If the rudder is in the longitudinal plane the aeroplane is symmetrical. By steady flight of such an aeroplane we mean that this plane is vertical, and that the C.G. of the aeroplane moves with a constant velocity in this plane, whilst there is no rotation. At present we take the air to be at rest, so that the motion through the air means the velocity of the aeroplane.

We define *normal flight* to be that steady flight near the earth's surface, in which the elevator is neutral, the propeller axis is horizontal, and the velocity of the machine is along the propeller axis, that is also horizontal (Fig. 1). The angle of attack is now a definite angle determined by the construction of the machine : we call it  $\alpha$ . Let  $W$  be the weight of the machine,  $U_0$  the velocity in normal flight, whilst  $R_1$  is the wing pressure and  $R_2$  the body resistance in the normal flight.

$R_1$  can be split up into a vertical component or *lift*  $R_1 \cos \alpha$ , and a horizontal component or *drag*  $R_1 \sin \alpha$ . Adding to the latter the body resistance or *body drag*  $R_2$ , we get a total drag  $R_1 \sin \alpha + R_2$ . But in steady flight all the forces acting on the machine must balance : hence the lift must balance the weight of the machine, whilst the drag must be overcome by a propeller thrust, which we call  $T$ . Thus

$$W = R_1 \cos \alpha, \quad T = R_1 \sin \alpha + R_2 = W \tan \alpha + R_2. \quad \dots \quad (1)$$

We can suppose that  $R_2$  is given for the normal motion, with the air-density near the earth's surface. To find  $R_1$ , we have  $R_1 = KSU_0^2 \sin \alpha$ , where  $S$  is the total area of the wings and  $K$  is a constant which can be determined experimentally for the particular type of wings used, and the air-density near the earth's surface. Hence

$$W = KSU_0^2 \sin \alpha \cos \alpha. \quad \dots \quad (2)$$

We can take  $W$  and  $U_0$  as prescribed, whilst  $K$  and  $\alpha$  would be determined by considerations of design which do not concern us here. The necessary wing area is given by (2).

The necessary propeller thrust is given by (1). In practice what we really need to know is the necessary power. If  $W$  and  $R_2$  are measured in lbs. wt. and  $U_0$  in mls./hr., the necessary horse-power for normal flight is at once seen to be

$$H_0 = \frac{WU_0 \tan \alpha}{375} + \frac{R_2 U_0}{375} \equiv H_1 + H_2. \quad \dots \dots \dots (3)$$

For reasons that will be evident later, we represent  $H_0$  as the sum of two parts,  $H_1$  ( $\equiv WU_0 \tan \alpha / 375$ ) being the H.P. required to overcome the drag due to the wings, and  $H_2$  ( $\equiv R_2 U_0 / 375$ ) being the H.P. required to overcome the body drag.

**3. Numerical Example.**—Suppose that  $W = 2000$  lbs.,  $U_0 = 60$  mls./hr.,  $\alpha = 6^\circ$ ,  $K = 0.0135$ , meaning that  $R_1 = 0.0135 S U_0^2 \sin \alpha$ , where  $R_1$  is in lbs. wt.,  $S$  in sq. ft.,  $U_0$  in mls./hr. We get

$$S = \frac{2000}{0.0135 \times (60)^2 \times \sin 6^\circ \cos 6^\circ} = \frac{4000}{1.35 \times 36 \times 0.2079} = 396 \text{ sq. ft.}$$

This means that each wing is to have an area of nearly 200 sq. ft., and using the customary ratio 6 for *span/chord*, it follows that each wing should be  $34\frac{1}{2}$  ft. in span by  $5\frac{1}{2}$  ft. in chord. The exact shapes, relative sizes and relative positions of the wings are matters for technical aeroplane design.

We also find  $H_1 = \frac{2000 \times 60 \times \tan 6^\circ}{375} = 33.6$  H.P.

Suppose that it is known that  $R_2 = 120$  lbs. wt.; we get

$$H_2 = \frac{120 \times 60}{375} = 19.2 \text{ H.P.}$$

Thus the necessary power for normal flight is  $H_0 = 52.8$  H.P.

This does not mean that in order to make a useful aeroplane it is only necessary that the engine shall be able to develop 52.8 H.P. We must allow for the imperfect efficiency of the propeller. We must also have a reserve of power in order to render possible flights other than normal flight, such as climbing, circling, etc. In practice the net available H.P. is not constant: we shall, however, adopt the approximate hypothesis that the net available H.P. for constant air-density is constant, except for low velocities, when the efficiency sinks very rapidly. We shall denote this by the symbol  $H^P$ . In the numerical example which we use in this paper, we shall take  $H^P = 100$ , which means an actual brake H.P. of about 140, except for low velocities.

**4. Climb and Descent with Neutral Elevator; Glide and Gliding Angle.**—Suppose that the H.P. used is different from  $H_0$ , whilst the elevator is neutral. Since the elevator is neutral the motion must be along the propeller axis, otherwise there would be a pressure on the tail and a consequent rotation. Hence the steady motion must be a climb or a descent, with the nose of the machine as defined by the propeller axis pointing along the direction of motion. Common sense tells us that an increased H.P. will give a climb, and a decreased H.P. will give a descent. To find the relation between the H.P. and the angle of climb or descent, we proceed thus.

Let the propeller axis point in a direction  $\theta$  above the horizontal (Fig. 2(a)), so that in a descent  $\theta$  is negative. If  $U$  is the velocity, the wing pressure

becomes  $\left(\frac{U}{U_0}\right)^2 R_1$ , since the angle of attack is unaltered; the body drag becomes  $\left(\frac{U}{U_0}\right)^2 R_2$ . For statical equilibrium we must have

$W \cos \theta$  = component of wing pressure perpendicular to the propeller axis.

$$T = W \sin \theta + \text{component of wing pressure along the propeller axis} \\ + \text{body drag.}$$

$$\text{Hence } W \cos \theta = \left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha, \quad T = W \sin \theta + \left(\frac{U}{U_0}\right)^2 R_1 \sin \alpha + \left(\frac{U}{U_0}\right)^2 R_2.$$

By (1),  $W = R_1 \cos \alpha$ ; hence  $U/U_0 = \cos^{\frac{1}{2}} \theta$ . Calculating  $TU/375$ , we find

$$H = \left( \frac{H_1}{\tan \alpha} \sin \theta + H_0 \cos \theta \right) \cos^{\frac{1}{2}} \theta. \quad \dots \dots \dots \quad (4)$$

If  $H$  is given,  $\theta$  is determined by equation (4);  $U$  can then be found, and the rate of climb or descent calculated.

It is useful to treat (4) numerically and graphically. Using the data of § 3, we get  $H = 324 \sin(\theta + 9^\circ 23') \cos^{\frac{1}{2}} \theta$ . In Fig. 2 (b) the curve marked  $W = 2000$  represents  $H$  plotted against  $\theta$ : the useful part of the curve is practically a straight line. We deduce the following results. When  $\theta = -9^\circ 23'$  the necessary power is zero. This means that if the engine and propeller are adjusted so as to give zero propeller thrust, the machine's steady motion is at an angle  $9^\circ 23'$  below the horizontal. We call this a *glide*, and the angle is the *gliding angle* for the given machine with neutral elevator. For steady flight in any direction above the glide, a positive thrust, and therefore positive H.P., is required, increasing from zero at the glide to 52.8 for the horizontal (normal) flight. If the aeroplane is to climb, still more power is required. Since  $H^P = 100$ , we see that the greatest angle of climb is about  $8\frac{1}{2}^\circ$ , and the rate of climb is then  $60 \sin 8\frac{1}{2}^\circ \sqrt{\cos 8\frac{1}{2}^\circ}$  mls./hr., i.e. nearly 780 ft./min. This is the *greatest rate of climb with neutral elevator*.

In order to get a steady descent steeper than the glide, the propeller thrust must be negative, giving a negative power. This can be done, the case of zero propeller thrust being given by a special state of the engine and propeller.

Equation (4) gives the gliding angle directly, since  $H = 0$  when

$$\tan \theta = -\frac{H_0}{H_1} \tan \alpha.$$

The gliding angle is therefore  $\tan^{-1} \left( \frac{H_0}{H_1} \tan \alpha \right)$ . This angle is of great importance in practice for several reasons, one being that it is accepted as an indication of the usefulness of the machine. From the mechanical point of view it is also of special interest and can be easily obtained as follows. If there is no propeller thrust, then the resultant of the wing pressure and body resistance must just balance the weight of the machine: this resultant must therefore be vertical. Now the wing pressure is  $\left(\frac{U}{U_0}\right)^2 R_1$ , making an angle  $90^\circ - \alpha$  with the propeller axis backwards; the body resistance is  $\left(\frac{U}{U_0}\right)^2 R_2$  along the propeller axis backwards. Hence the resultant makes with the backward direction of the propeller axis an angle

$$\cot^{-1} \frac{\left(\frac{U}{U_0}\right)^2 R_1 \sin \alpha + \left(\frac{U}{U_0}\right)^2 R_2}{\left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha} = \cot^{-1} \frac{R_1 \sin \alpha + R_2}{R_1 \cos \alpha} \\ = \cot^{-1} \left( \frac{W \tan \alpha + R_2 \tan \alpha}{W \tan \alpha} \right) = \cot^{-1} \left( \frac{H_0}{H_1} \tan \alpha \right)$$

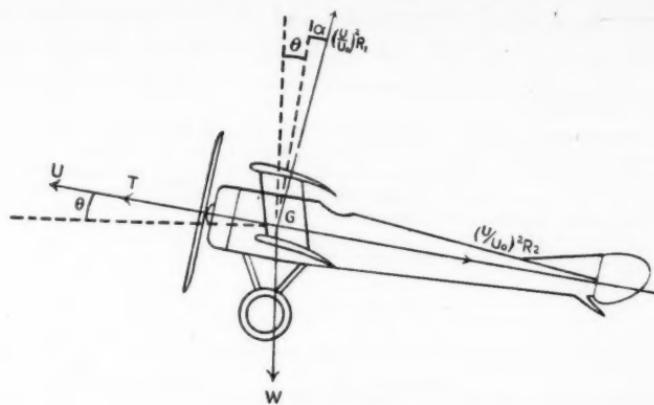


FIG. 2 (a).

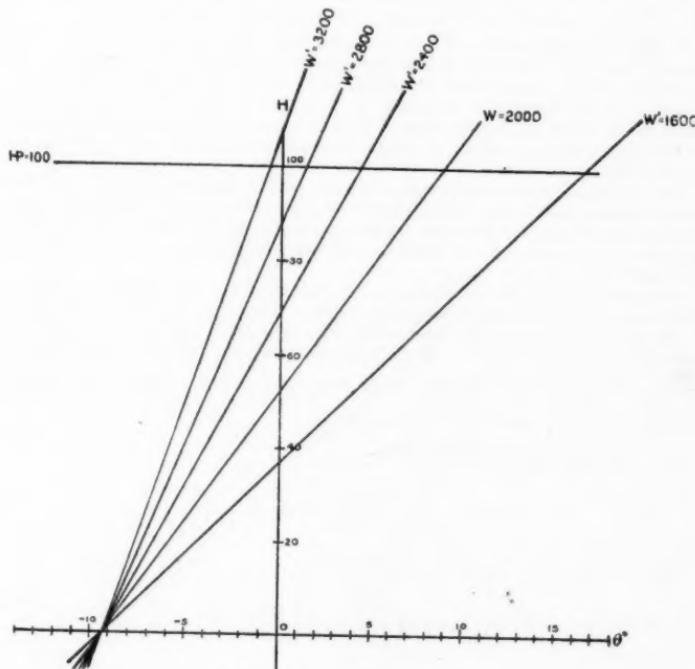


FIG. 2 (b)

by (3). The resultant therefore makes with the forward direction of the propeller axis an angle  $90^\circ + \tan^{-1} \left( \frac{H_0}{H_1} \tan \alpha \right)$ , and as this resultant must be vertical, it follows that the propeller axis is at an angle  $\tan^{-1} \left( \frac{H_0}{H_1} \tan \alpha \right)$  below the horizontal.

**5. Approximate method.**—The fact that the curve in Fig. 2 (b) is practically a straight line suggests that for moderate values of  $\theta$ —and in ordinary practice only such angles are of importance—we can approximate, by putting  $\cos \theta = 1$ . We get  $U = U_0$  practically, so that the speed is unaffected by any change in  $H$  so long as the elevator is neutral. We also get

$$H = H_0 + \frac{W U_0 \sin \theta}{375}$$

$$\text{or} \quad H = H_0 + H_1 \frac{\theta}{\alpha}.$$

Hence the greatest angle of climb with neutral elevator is  $\frac{H^P - H_0}{H_1} \alpha$ , and the greatest rate of climb is  $\frac{375(H^P - H_0)}{W}$  mls./hr., i.e.  $\frac{33000(H^P - H_0)}{W}$  ft./min. It will be seen that the approximate results agree with those found graphically, in § 4. The gliding angle is given by  $H = 0$ , i.e.  $0 = -\frac{H_0}{H_1} \alpha$ .

We get the important approximate rule: *the rate of climb is determined by the net power available over and above that required for normal flight.*

**6. Use of the Elevator.**—We have so far taken the elevator as fixed in its neutral position. The fact is that the climb and descent of an aeroplane are functions of the engine and propeller, and not of the elevator. The main function of the elevator in aeroplane performance is not to raise or lower the path of the machine, but to raise or lower the nose of the machine relatively to the direction of motion: in other words, *the elevator is used to change the angle of attack*.

By hypothesis the wing pressure always passes through the C.G., and the same is assumed of the body resistance. It follows that in steady flight the motion through the air must always be parallel to the plane of the elevator. This is by no means true in practice, but the simplification will serve for our purposes.

Suppose now that in horizontal steady flight the elevator makes an angle  $\beta$  with the neutral position, measured upwards. The propeller axis will therefore be at an angle  $\beta$  above the horizontal, Fig. 3 (a), and the angle of attack is therefore  $\alpha + \beta$  instead of  $\alpha$ . If the horizontal velocity is  $U$ , the wing pressure is  $\left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1$ , and the body drag is  $\left( \frac{U}{U_0} \right)^2 R_2$ . Hence for steady motion we get the conditions of resolution :

$$W \cos \beta = \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos \alpha,$$

$$T = W \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin \alpha + \left( \frac{U}{U_0} \right)^2 R_2.$$

$$\text{But } W = R_1 \cos \alpha: \text{ hence } \frac{U}{U_0} = \sqrt{\frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)}} = \sqrt{\frac{\tan \alpha}{\tan \alpha + \tan \beta}}. \text{ Also}$$

$$T = W \sin \beta + W \cos \beta \tan \alpha + \left( \frac{U}{U_0} \right)^2 R_2 = \frac{W \sin(\alpha + \beta)}{\cos \alpha} + \left( \frac{U}{U_0} \right)^2 R_2$$

$$= W \tan \alpha \cos \beta \left( \frac{U_0}{U} \right)^2 + R_2 \left( \frac{U}{U_0} \right)^2$$

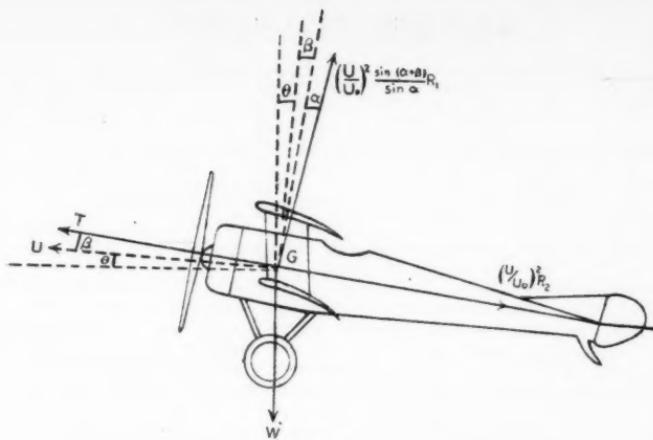


FIG. 3 (a).

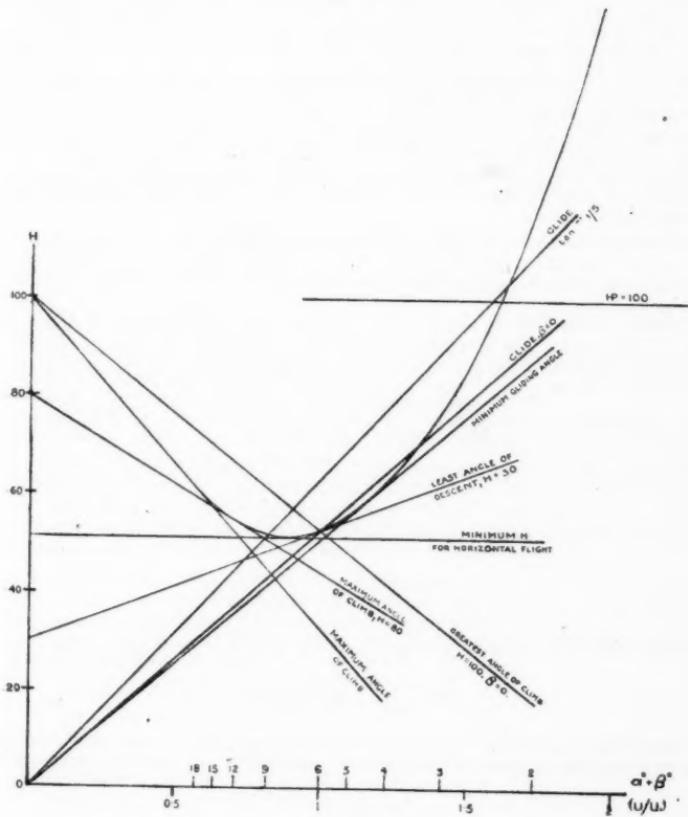


FIG. 3 (b).

To get the necessary power we multiply  $T$  by  $U \cos \beta/375$ , i.e. by

$$(U_0 \cos \beta/375)(U/U_0).$$

We get 
$$H = H_1 \cos^2 \beta \left( \frac{U_0}{U} \right)^2 + H_2 \cos \beta \left( \frac{U}{U_0} \right)^3, \quad \left. \begin{array}{l} \\ \end{array} \right\} \dots \dots \dots (5)$$

with 
$$\left( \frac{U_0}{U} \right)^2 = 1 + \frac{\tan \beta}{\tan \alpha}, \text{ i.e. } \beta = \tan^{-1} \left[ \left( \frac{U_0}{U} \right)^2 - 1 \right] \tan \alpha. \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

It must be remembered, however, that equations (5) can be approximately true only for moderate angles of attack, in view of the assumptions as to air forces, laid down in § 1. In practice, the angle of attack must for reasons of safety be less than a certain maximum; in our case a safe angle of attack would be something less than 18 or 20 degrees for the type of wing assumed in § 3. Taking an upper limit of  $18^\circ$  for the angle of attack, we have an upper limit of  $12^\circ$  for  $\beta$ . We are therefore at liberty to approximate in (5), since the lower limit of  $\beta$  is obviously  $-6^\circ$ . We have the simplified equations

$$H = H_1 \left( \frac{U_0}{U} \right)^2 + H_2 \left( \frac{U}{U_0} \right)^3, \quad \beta = \left[ \left( \frac{U_0}{U} \right)^2 - 1 \right] \alpha, \quad \frac{U}{U_0} = \sqrt{\frac{\alpha}{\alpha + \beta}}. \quad \dots \dots \dots (6)$$

In order then to have horizontal flight with  $U > U_0$ ,  $\beta$  must be negative and the elevator must be turned down; to have  $U < U_0$ ,  $\beta$  must be positive and the elevator must be turned up. In addition the engine and propeller must be made to give the power indicated by (6).

The values of  $H$ , plotted against  $U/U_0$ , are shown in Fig. 3 (b), and the corresponding angles of attack are indicated. It is at once seen that the greatest possible velocity for horizontal flight is given by  $U/U_0 = 1.6$ , so that  $U$  is about 96 mls./hr., with angle of attack  $23^\circ$ . The minimum horizontal velocity, with angle of attack  $18^\circ$ , would be nearly 35 mls./hr., but the complex question of the efficiency of the propellér enters here, and this result is certainly not reliable.

**7. General Problem.**—In § 6 it was supposed that the engine is adjusted so as to give just the necessary power for horizontal flight. If, however, the actual H.P. differs from that prescribed by equation (6) we get a climb or a descent.

Let the elevator be at angle  $\beta$ , and suppose that the direction of flight is at an angle  $\theta$  above the horizontal, so that the angle of attack is  $\alpha + \beta$ , and the propeller axis makes an angle  $\beta + \theta$  with the horizontal. For statical equilibrium we now get

$$W \cos(\beta + \theta) = \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos \alpha,$$

$$T = W \sin(\beta + \theta) + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin \alpha + \left( \frac{U}{U_0} \right)^2 R_2.$$

As  $W = R_1 \cos \alpha$ , we find  $\frac{U}{U_0} = \sqrt{\frac{\sin \alpha \cos(\beta + \theta)}{\sin(\alpha + \beta)}}$ , whilst

$$T = W \frac{\sin(\alpha + \beta + \theta)}{\cos \alpha} + \left( \frac{U}{U_0} \right)^2 R_2.$$

Approximating, we find  $\frac{U}{U_0} = \sqrt{\frac{\alpha}{\alpha + \beta}}$ , i.e.  $\beta = \left[ \left( \frac{U_0}{U} \right)^2 - 1 \right] \alpha$ , 
$$\left. \begin{array}{l} \\ \end{array} \right\} \dots \dots \dots (7)$$

$$H = \frac{WU\theta}{375} + H_1 \left( \frac{U_0}{U} \right)^2 + H_2 \left( \frac{U}{U_0} \right)^3.$$

Comparing (7) with (6) we at once get the approximate rule: *the rate of climb with angle of attack  $\alpha + \beta$  is determined by the net power available over*

and above that required for horizontal flight with this angle of attack. In Fig. 3 (b) the rate of climb for any angle of attack is given by the intercept of the corresponding ordinate between the  $H$  curve and the  $HP$  line.

To get the maximum rate of climb we have to find the maximum intercept. The figure shows that for the machine of § 3, the maximum rate of climb occurs for angle of attack  $8^\circ$  and  $U/U_0 = 0.87$ , so that  $U = 52$  mls./hr., giving a rate of climb 805 ft./min. We can also calculate this result by finding the minimum value of  $H$  in (6). Using the calculus, we see that  $H$  is a minimum when  $U/U_0 = \sqrt[4]{H_1/3H_2}$ , giving  $H = \frac{1}{3}H_1 \sqrt[4]{3H_2/H_1}$ . The maximum rate of climb is therefore  $\frac{33000}{W} (HP - \frac{1}{3}H_1 \sqrt[4]{3H_2/H_1})$  ft./min. Substituting

$$H_1 = 33.6, \quad H_2 = 19.2, \quad HP = 100,$$

we verify the results obtained graphically.

The maximum rate of climb does not necessarily correspond to normal flight, which means to neutral elevator: this happens only when  $H_1 = 3H_2$ .

A descent is calculated in the same way,  $\theta$  being negative and  $H$  less than that required for horizontal flight with the same angle of attack.  $H = 0$  gives the glide for elevator  $\beta$ . We at once find that the gliding angle is approximately

$$375 \left[ \left( \frac{U_0}{U} \right) H_1 + \left( \frac{U}{U_0} \right)^3 H_2 \right].$$

Using  $(U/U_0)^2 = \alpha/(\alpha + \beta)$ , we find that the gliding angle is

$$\alpha + \beta + \frac{\alpha^2}{\alpha + \beta} \frac{H_2}{H_1}. \quad \dots \dots \dots \quad (8)$$

Using the more accurate method by equating  $T$  above to zero, we find the gliding angle to be

$$\beta + \tan^{-1} \left[ 1 + \frac{\sin \alpha}{\sin(\alpha + \beta)} \frac{H_2}{H_1} \right] \tan \alpha, \quad \dots \dots \dots \quad (9)$$

and this agrees with (8) when  $\alpha/(\alpha + \beta)$  is small. The results can be readily verified by the method of § 4. Notice that  $\beta = 0$  gives the value  $\frac{H_0}{H_1} \alpha$  found in § 4.

In order that an ordinary glide be possible it is obvious from (8) that  $\alpha + \beta$  must be positive, giving a justification for the principle of the *longitudinal dihedral*, which is of such repute with designers of machines.

For different positions of the elevator we get different gliding angles, with zero thrust. There is a minimum gliding angle, and easy algebra or calculus tells us that the angle (8) is a minimum when  $\alpha + \beta = \alpha \sqrt{H_2/H_1}$ , whilst the minimum gliding angle itself is  $2\alpha \sqrt{H_2/H_1}$ . The direction of zero lift bisects the angle between the horizontal and the direction of the glide.

The machine of § 3 has a minimum gliding angle of just over  $9^\circ$ , with angle of attack  $4\frac{1}{2}^\circ$ , so that  $\beta = -1\frac{1}{2}^\circ$ ; the velocity is then 69 mls./hr.

8. It is obvious from (8) that for a given gliding angle there are two values of  $\beta$ , so that two configurations of the machine are possible. Thus, suppose that with the machine in § 3 it is desired to glide down from a height of 1 mile to a point distant 5 miles horizontally. The angle of glide is  $\tan^{-1} \frac{1}{5}$ , i.e.  $11\frac{1}{3}^\circ$ . The equation for  $\beta$  in degrees is

$$\alpha + \beta + \frac{36}{\alpha + \beta} \frac{19.2}{33.6} = 11\frac{1}{3},$$

giving:  $\alpha + \beta = 9^\circ$  and  $U = 49$  mls./hr. or  $\alpha + \beta = 24^\circ$  and  $U = 98$  mls./hr. Each solution has its advantages: the first gives a slow descent, but the second is safer because of the small angle of attack.

9. Now take the following problem: *Given  $H$  and  $\theta$ , is steady flight possible, and if so what angle of attack is required?* Equations (7) give, since

$$H_1 = WU_0 \tan \alpha / 375,$$

$$H = H_1 \frac{\theta}{\alpha} \left( \frac{U}{U_0} \right) + H_1 \left( \frac{U_0}{U} \right)^3 + H_2 \left( \frac{U}{U_0} \right)^3.$$

This is a biquadratic in  $U/U_0$ , and when it is solved we have  $\sqrt{\alpha/(\alpha+\beta)}$ , so that  $\beta$  is known. It is easier, however, to discuss the equation graphically, by plotting the curves with ordinates

$$H_1 \left( \frac{U_0}{U} \right) + H_2 \left( \frac{U}{U_0} \right)^3, \quad H - H_1 \frac{\theta}{\alpha} \left( \frac{U}{U_0} \right)$$

respectively. The first curve is the one already plotted in Fig. 3 (b) (with the data of § 3). The second is a straight line which cuts the  $H$  axis at the prescribed value  $H$ , and whose gradient is  $-\frac{\theta}{\alpha} H_1$ , the gradient being measured by the number of units of  $H$  compared to the number of units of  $U/U_0$ . If the line cuts the curve the intersections give (in general) the two alternative velocities and elevator positions. If the line does not cut the curve, no steady flight is possible. For any given value of  $H$  there is a limiting value of  $\theta$  above which steady flight is impossible: this limiting value is obtained by drawing through the point  $H$  on the  $H$ -axis the tangent to the curve; if this tangent has gradient  $g$  with the velocity axis, then the limiting angle is  $-\frac{g}{H_1} \alpha$ , in degrees if  $\alpha$  is in degrees. If  $g$  is positive the result gives the least angle of descent, if negative the greatest angle of climb, with the prescribed  $H$ . The gradient of the tangent through the origin determines the minimum gliding angle.

In Fig. 3 (b) we have  $H_1 = 33.6$ ,  $\alpha = 6^\circ$ ; also for the tangent through the origin  $g = 50.5$ ; hence the minimum gliding angle is just over  $9^\circ$ ;  $(U/U_0)$  is given by the point of contact as 1.15, so that  $U = 69$  mls./hr. To find the elevator positions and velocities for a gliding angle of say  $11\frac{1}{2}^\circ$ , we draw through the origin the line whose gradient is given by  $\frac{g}{33.6} = 11\frac{1}{2}$ , i.e.  $g = 63.4$ .

This cuts the curve at  $U/U_0 = 0.81$  and at  $U/U_0 = 1.63$ , so that  $U = 49$  mls./hr.,  $\alpha + \beta = 9^\circ$ , and  $U = 98$  mls./hr.,  $\alpha + \beta = 2\frac{1}{2}^\circ$  respectively, as found in § 8. The line joining the origin to the point on the curve corresponding to the greatest permissible value of the angle of attack, viz.  $\alpha + \beta = 18^\circ$ , has gradient  $g = 108$ , giving gliding angle  $19\frac{1}{2}^\circ$ ; up to this gliding angle two possible angles of attack exist; beyond this only one exists. The gliding angle for neutral elevator is given by  $g = 52.8$ ; the figure shows that the same gliding angle is obtained with  $U/U_0 = 1.33$ , so that  $\alpha + \beta = 3\frac{1}{2}^\circ$  and  $\beta = -2\frac{1}{2}^\circ$ .

For any value of  $H$  less than the minimum found in § 7, i.e. 51.2 for the data of § 3, the steady flight must be a descent; thus for  $H = 30$ , we get for the tangent to the curve,  $g = 22.5$ , so that the least angle of descent is just about  $4^\circ$ ; the velocity is about 58 mls./hr. and the elevator nearly neutral. To get an angle of descent of say  $11\frac{1}{2}^\circ$ , we have  $g = 63.4$ ; this is only possible for a high velocity of 116 mls./hr., since the other intersection with the curve would give an angle of attack greater than the safe maximum; the same applies to all angles of descent with  $g$  greater than 54.5, i.e. angles of descent greater than  $9\frac{1}{2}^\circ$ . For angles of descent lying between  $4^\circ$  and  $9\frac{1}{2}^\circ$  two possible angles of attack exist, with corresponding velocities.

$H = 51.2$ , the minimum for horizontal flight, gives the result that no climbing is possible, but the machine can just fly horizontally, or descend. For angles of descent less than  $3\frac{1}{2}^\circ$  two configurations exist; for angles of descent greater than  $3\frac{1}{2}^\circ$  only one angle of attack is possible, with large velocity.

When  $H > 51.2$ , climbing is possible. Thus let  $H = 80$ ; the tangent to the curve gives  $g = -36$ , so that the greatest angle of climb is nearly  $6\frac{1}{2}^\circ$ , with a velocity 44 mls./hr. For an angle of climb less than  $6\frac{1}{2}^\circ$  and greater than  $5\frac{1}{2}^\circ$ , two configurations are possible; for a more moderate climb or for a descent only one configuration is possible, so as to avoid unsafe angles of attack. Horizontal flight with  $H = 80$  requires  $U = 86$  mls./hr. and  $\beta = -3\frac{1}{4}^\circ$ .

Taking the greatest possible value of  $H$  we see that the greatest possible angle of climb is given by  $g = -67$ , and the angle is  $12^\circ$ . This, of course, does not give the maximum rate of climb, since the velocity is small; it corresponds to  $\beta = 12^\circ$ . Taking  $\beta = 0$  we at once get that the greatest angle of climb for neutral elevator is  $8\frac{1}{2}^\circ$ , as found in § 4.

Negative values of  $H$  can also be treated in this way.

10. Diving and Upside-Down Flight, with Zero Propeller Thrust.—The method of § 9 applies only to small inclinations of the path. A climb is never

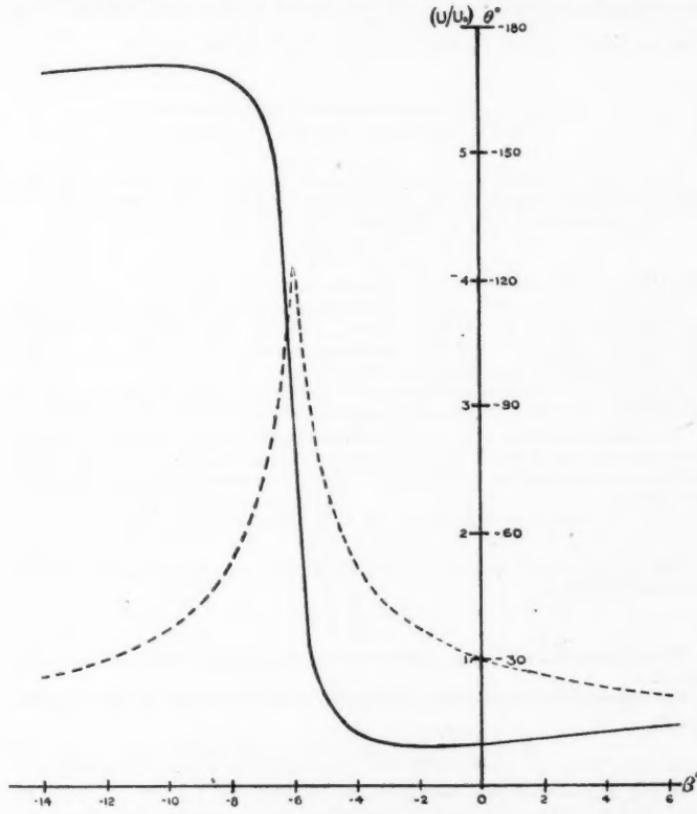


FIG. 4.

at a large angle with the horizontal, but a descent may be steep, amounting to what is called a dive. The general theory of a dive involves a study of

propellers; we shall therefore consider only dives in which the propeller thrust is zero, in other words steep glides. As we now have to consider large values of  $\theta$  we cannot use the approximations of § 7, equations (7) and (8), and the curve of Fig. 3 (b) is no longer applicable. Equation (9), however, is applicable. In fact we can use it in the form

$$\theta = -\beta - \tan^{-1} \left( 1 + \frac{H_2}{H_1} \frac{\alpha}{\alpha + \beta} \right) \tan \alpha. \quad (10)$$

In order to get concrete results we plot  $\theta$  against  $\beta$  in Fig. 4; the corresponding ratio  $U/U_0$  is also indicated, this being given by

$$U/U_0 = \sqrt{\sin \alpha \cos(\beta + \theta) / \sin(\alpha + \beta)},$$

as found in § 7 before the approximation was made. Since  $\alpha, \beta$  are small, we can use  $U/U_0 = \sqrt{\alpha \cos(\beta + \theta) / (\alpha + \beta)}$ . There is no particular difficulty in getting the numerical values of  $U/U_0$ , except for the case  $\alpha + \beta = 0$ . Using

(10) we find  $\cos(\beta + \theta) = 1 / \sqrt{1 + \left( 1 + \frac{H_2}{H_1} \frac{\alpha}{\alpha + \beta} \right)^2 \tan^2 \alpha}$ ; hence

$$\frac{U}{U_0} = \sqrt{\frac{\alpha}{\sqrt{(\alpha + \beta)^2 + \left( \alpha + \beta + \frac{H_2}{H_1} \alpha \right)^2 \tan^2 \alpha}}}.$$

Thus when  $\alpha + \beta = 0$  we have  $U/U_0 = \sqrt{H_1/H_2} \tan \alpha$ , so that for the aeroplane of § 3 the velocity ratio  $U/U_0$  becomes 4.08. The maximum value of  $U/U_0$  occurs when  $\alpha + \beta = -H_2 \alpha \sin^2 \alpha / H_1$ , and the value is

$$U/U_0 = \sqrt{H_1/H_2} \sin \alpha,$$

but this is hardly distinguishable from the case just discussed.

$\theta = -10^\circ, -20^\circ, -30^\circ, \dots$  are glides of increasing steepness till  $\theta = -90^\circ$  is a vertical dive;  $\theta = -100^\circ, -110^\circ, \dots$  are at once seen to be glides of decreasing steepness with the aeroplane upside down. For the vertical dive the velocity is something like 246 mls./hr.

It is clear, of course, that these results are only a rough presentation of the actual facts, but they do indicate phenomena of interest and importance.

**11. Load Variations.**—In practical flight the load is not constant: we shall now discuss the effects of varying load, assuming that the c.g. is unchanged. If the new weight of the machine is  $W'$ , then, for normal flight with velocity  $U$ , we have

$$W' = \left( \frac{U}{U_0} \right)^2 R_1 \cos \alpha, \quad T = \left( \frac{U}{U_0} \right)^2 R_1 \sin \alpha + \left( \frac{U}{U_0} \right)^2 R_2.$$

But  $W = R_1 \cos \alpha$ ; hence  $U/U_0 = \sqrt{W'/W}$ , and we at once deduce that the necessary power is

$$H = \left( \frac{W'}{W} \right)^{\frac{3}{2}} H_0.$$

Thus increased load means greater power for normal flight, but increased speed.

For climbing and descending flight with neutral elevator we have, instead of (4),

$$H = \left( \frac{W'}{W} \right)^{\frac{3}{2}} \left( \frac{H_1}{\tan \alpha} \sin \theta + H_0 \cos \theta \right) \cos^{\frac{1}{2}} \theta. \quad (11)$$

In Fig. 2 (b) curves have been added for  $W'/W = 0.8, 1.2, 1.4, 1.6$ , using the data of § 3. When the load is diminished the possibility for climbing is increased: thus for  $W' = 1600$  lbs. the greatest angle of climb with neutral elevator is  $16\frac{1}{2}^\circ$ , and the rate of climb is nearly 1300 ft./min. If the load is increased, the possibility for climbing diminishes: thus for  $W' = 2400$  lbs.,

the greatest angle of climb is  $4\frac{1}{2}^\circ$  and the rate of climb is 420 ft./min. With  $W' = 2800$  lbs., the greatest angle of climb is  $1\frac{1}{4}^\circ$  and the rate of climb is not appreciably more than 140 ft./min. With  $W' = 3200$  lbs. horizontal flight is impossible with neutral elevator: in fact, the only possible flight with neutral elevator is a descent at an angle at least  $\frac{3}{4}^\circ$ .

To find the greatest load for which normal flight is just possible, we put

$$HP = \left(\frac{W'}{W}\right)^{\frac{2}{3}} H_0.$$

Hence in our case the greatest load is  $2000 \left(\frac{100}{52.8}\right)^{\frac{2}{3}} = 3062$  lbs.

It is of interest to note that the gliding angle is unaffected by the loading, but of course the rate of descent will vary as the square root of the load.

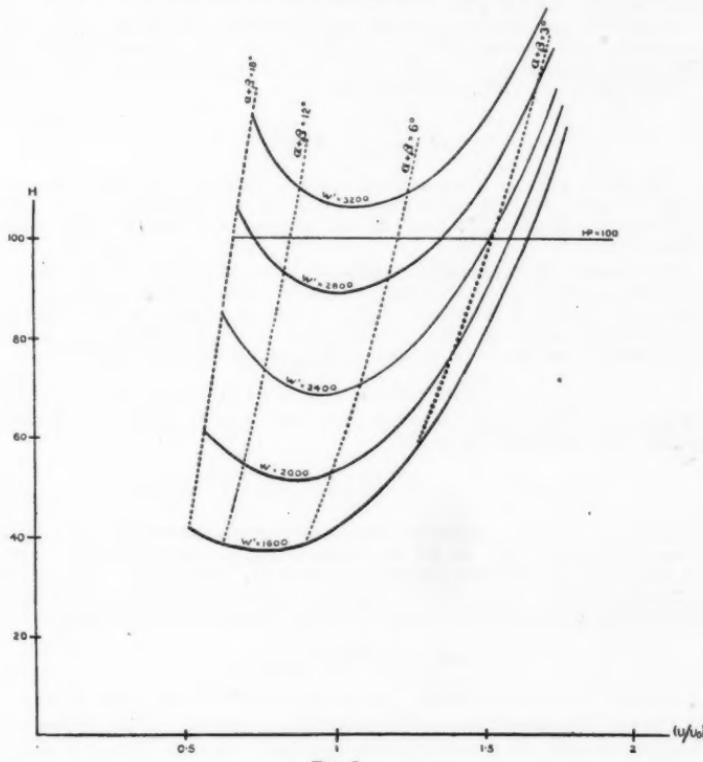


FIG. 5.

For general horizontal flight with angle of attack  $\alpha + \beta$ , we easily show that

$$H = \left(\frac{W'}{W}\right)^{\frac{2}{3}} H_1 \left(\frac{U_0}{U}\right) + H_2 \left(\frac{U}{U_0}\right)^{\frac{2}{3}}, \quad (12)$$

where  $U/U_0 = \sqrt{\alpha W' / (\alpha + \beta) W}$ , assuming that the angle of attack remains small. In Fig. 5 we have plotted  $H$  against  $(U/U_0)$ , for  $W'/W = 0.8, 1, 1.2, 1.4, 1.6$ .

The corresponding angles of attack are indicated by means of dotted curves: they are really

$$H = \left[ \left( \frac{\alpha + \beta}{\alpha} \right)^2 H_1 + H_2 \right] \left( \frac{U}{U_0} \right)^3.$$

Taking now the general problem, we see that for equations (7) we get

$$H = \frac{W'U_0}{375} + \left( \frac{W'}{W} \right)^2 H_1 \left( \frac{U_0}{U} \right) + H_2 \left( \frac{U}{U_0} \right)^3, \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad (13)$$

where  $\frac{U}{U_0} = \sqrt{\frac{\alpha}{\alpha + \beta} \frac{W'}{W}}.$

It follows that all the information given by Fig. 3 (b) for load 2000 lbs. is given by Fig. 5 for any load we wish to take, assuming that sufficient curves have been drawn. (See also § 18.)

To get the absolutely maximum load for which horizontal flight (not necessarily with neutral elevator) is possible, we can interpolate from the figure, or, analytically, we find  $W'/W$  so that the minimum value of  $H$  in (12) is just equal to  $H^P$ . We get the equation

$$H^P = \frac{4}{3} \left( \frac{W'}{W} \right)^{\frac{2}{3}} H_1 \sqrt{\frac{3H_2}{H_1}}.$$

The gliding angle for given angle of attack is unaffected by load variations. It is useful to note that the minima on the different curves in Fig. 5 lie on the curve  $H = 4H_2(U/U_0)^3$ : hence for all the minima we have the same angle of attack.

**12. Variation in Air-Density: Ceiling.**—The density of the air is different at different heights above the earth's surface: our calculations have so far been made for flight low down. Let  $\rho'$  be the air-density at any altitude,  $\rho$  the density at the ground; then we must introduce a factor  $\rho'/\rho$  in the wing-pressure and the body-resistance. For normal flight we find

$$(U/U_0) = \sqrt{\rho/\rho'}, \quad H = \sqrt{\rho/\rho'} \cdot H_0. \quad (14)$$

Thus, at an altitude above the ground, the speed must be greater and also the power. For equations (6) we get

$$\frac{U}{U_0} = \sqrt{\frac{\rho}{\rho'} \frac{\alpha}{\alpha + \beta}}, \quad H = \left( \frac{\rho}{\rho'} \right) H_1 \left( \frac{U_0}{U} \right) + \left( \frac{\rho}{\rho'} \right) H_2 \left( \frac{U}{U_0} \right)^3. \quad (15)$$

Using the fact that with an increase in altitude of 10,000 ft. the air-density changes to 74 per cent., we get the curves in Fig. 6, derived from the curve in Fig. 3 (b), for altitudes 10,000 and 20,000 ft. respectively. The curves are in fact similar curves with the origin as centre of similitude. The angles of attack are indicated by means of the dotted lines, whose equations are

$$\frac{H}{\left( \frac{U}{U_0} \right)} = \frac{\alpha + \beta}{\alpha} H_1 + \frac{\alpha}{\alpha + \beta} H_2.$$

The available power is also a function of the air-density. As an approximation we can make the available power vary as the density. We see from the figure that it is impossible to get any useful flight at 20,000 ft., and by interpolation we find that the greatest height at which flight is possible (in anything but a descent) is nearly 15,000 ft. This height is called the *ceiling* of the aeroplane considered.

We can get the result by calculation. If  $H^P$  is the greatest available power at density  $\rho$ , then at density  $\rho'$  we have available power  $\frac{\rho'}{\rho} H^P$ . If this is just

equal to the minimum value of  $\left(\frac{\rho'}{\rho}\right) H_1 \left(\frac{U_0}{U}\right) + \left(\frac{\rho'}{\rho}\right) H_2 \left(\frac{U}{U_0}\right)^3$ , we have the limiting value of  $\rho'/\rho$ , so that the ceiling can be calculated. The minimum value is easily seen to be  $\frac{1}{3} H_1 \sqrt{\frac{\rho}{\rho'}} \sqrt[4]{\frac{3H_2}{H_1}}$ . Hence the equation for  $\rho'/\rho$  is

$$\frac{\rho'}{\rho} \cdot HP = \frac{1}{3} H_1 \sqrt{\frac{\rho}{\rho'}} \sqrt[4]{\frac{3H_2}{H_1}},$$

so that

$$\frac{\rho'}{\rho} = \left( \frac{HP}{\frac{1}{3} H_1 \sqrt{\frac{3H_2}{H_1}}} \right)^{\frac{2}{3}}.$$

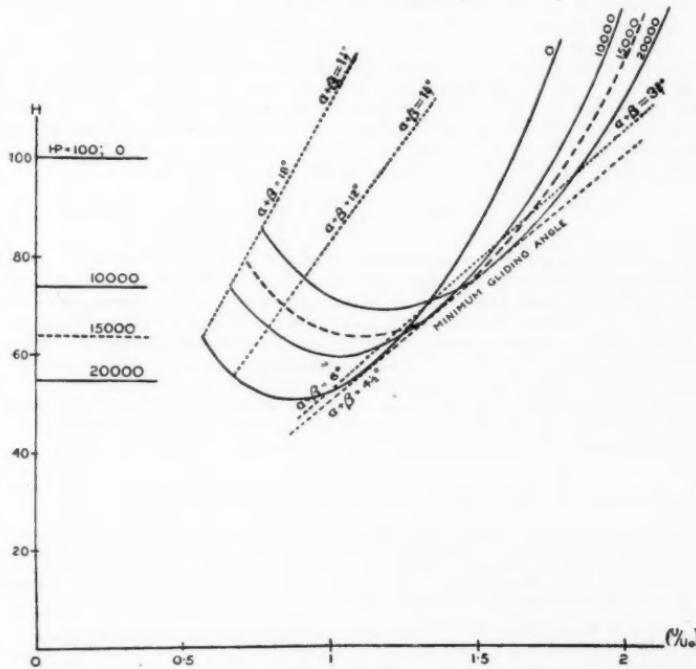


FIG. 6.

Let the ceiling be at height  $10000\lambda$  ft.; then  $\rho/\rho' = (0.74)^{-\lambda} = (1.352)^\lambda$ .

Hence

$$\lambda = \frac{2}{3} \frac{\log_{10} \left( \frac{HP}{\frac{1}{3} H_1 \sqrt{\frac{3H_2}{H_1}}} \right)^{\frac{2}{3}}}{\log_{10} 1.352},$$

and the height of the ceiling is approximately

$$51,000 \left[ \log_{10} HP - \log_{10} \left( \frac{1}{3} H_1 \sqrt{\frac{3H_2}{H_1}} \right)^{\frac{2}{3}} \right] \text{ ft.} \quad \dots \dots \dots \quad (16)$$

In our case this is 14,800 ft. The curve and the available power line for

about this height have been added in Fig. 6, and the calculation is seen to agree with the graph. (See also § 18.)

The effect of the loading on the height of the ceiling can be readily deduced: it is obvious that the ceiling is raised by diminishing the load, and *vice versa*. To deal with load and air-density together, we can use

$$U/U_0 = \sqrt{\frac{\rho}{\rho'}} \frac{W'}{W} \frac{\alpha}{\alpha + \beta}, \quad H = \frac{W' U_0}{375} + \left(\frac{\rho}{\rho'}\right) \left(\frac{W'}{W}\right)^2 H_1 \left(\frac{U_0}{U}\right) + \left(\frac{\rho}{\rho'}\right) H_2 \left(\frac{U}{U_0}\right)^3. \quad (17)$$

**13. Flying in a Steady Wind.**—If the air is not at rest but in motion in the form of a steady wind, we can find the steady flight of the aeroplane by calculating the motion relative to the air, and adding on the motion of the air: this is because the air-forces depend on the motion *through the air*, and not on the complete motion of the aeroplane as seen by an observer on the earth.

Let there be a steady horizontal wind with velocity  $U'$  (mls./hr.), this being reckoned positive in the direction that the aeroplane flies when in normal flight (to the left in our figures). For given angle of attack and power the motion *through the air* is determined completely by our former work. Let the velocity relative to the air be  $U$  (mls./hr.) in direction making an angle  $\theta$  with the horizontal (climb +, descent -). We at once get that the direction of flight as seen from the earth makes with the horizontal an angle

$$\tan^{-1} \left( \frac{U \sin \theta}{U \cos \theta + U'} \right). \quad (18)$$

For small  $\theta$  we get the approximate value  $U\theta/(U + U')$ . Positive  $U'$  gives diminished  $\theta$ , negative  $U'$  gives increased  $\theta$ : the former corresponds to flying with the wind, the latter to flying against the wind. Hence in order to cover as great a distance horizontally as possible, the pilot should fly with the wind; to climb as steeply as possible, he should fly against the wind. When descending or gliding, he should land against the wind so as to have a small velocity relative to the ground.

Now let there be a horizontal wind  $U'$ : if the pilot wishes to fly in a direction making an angle  $\Psi$  with the direction of the wind, he should point his machine so that it makes an angle  $\psi$  with the wind, where

$$\frac{U \sin \psi}{U' + U \cos \psi} = \tan \Psi, \text{ so that } \psi = \Psi + \sin^{-1} \left( \frac{U'}{U} \sin \Psi \right). \quad (19)$$

If a pilot at height  $h$  above the ground descends so that relative to the wind his angle of descent is  $\theta$  and velocity  $U$ , in a plane at angle  $\psi$  with the direction of the wind, then his rate of vertical fall will be approximately  $U\theta$ , whilst the horizontal motion as seen from the earth will be at speed

$$\sqrt{U^2 + 2UU' \cos \psi + U'^2} \text{ at angle } \tan^{-1} \{U \sin \psi / (U' + U \cos \psi)\}$$

with the direction of the wind. Hence the angle of descent can be put

$$\theta \left( 1 + 2 \frac{U'}{U} \cos \psi + \frac{U'^2}{U^2} \right)^{-\frac{1}{2}}$$

if  $U'/U$  is not too large. Let  $r_0 = h/\theta$ , which is the horizontal distance to reach the ground when  $U' = 0$ . The distance  $r$  with wind  $U'$  at angle  $\psi$  with the wind is

$$r = r_0 \left( 1 + 2 \frac{U'}{U} \cos \psi + \frac{U'^2}{U^2} \right)^{\frac{1}{2}}. \quad (20)$$

We can trace on the ground the locus of points reached in different directions  $\Psi$  given by (19) by using  $r$  from (20), the pole of the polar coordinates thus defined being the point on the earth vertically below the position of the machine in the air. For any given value of  $U'/U$  the locus is in fact a circle.

14. Circling Flight, Control, Side-Slip.—In order to obtain circular and still more general flight, a modern aeroplane is provided with a rudder, as already mentioned in § 1, and with *ailerons*, or *flaps* at the wings, Fig. 7.

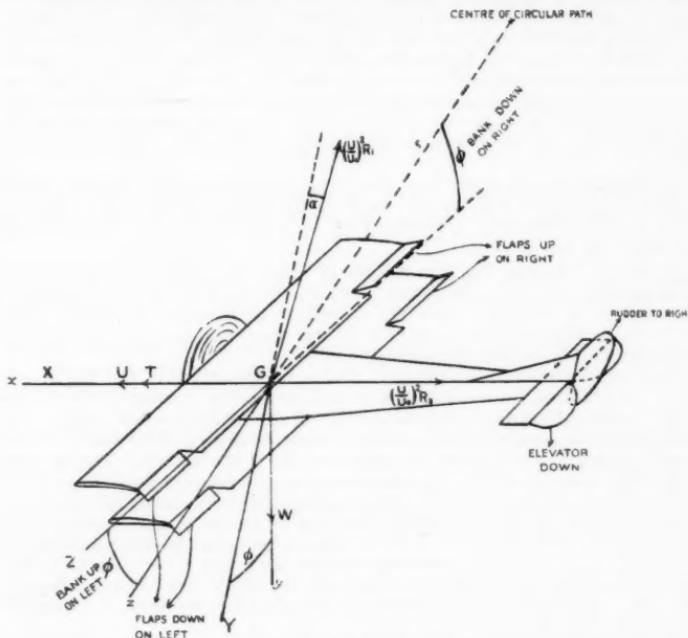


FIG. 7.

The pilot can turn the rudder and flaps from his seat, the latter being so arranged that when one flap is turned up, the other is turned down.

Suppose that an aeroplane is flying horizontally in still air and the pilot wishes to turn to the right. In order to supply the centripetal force to make the aeroplane leave its straight course, the machine is *banked* to the right, so that the wing-pressure gives a horizontal component which is to act as centripetal force. The pilot therefore makes the right-hand aileron turn up and the left-hand aileron down: the diminished angle of attack on the right gives diminished lift, and the increased angle of attack on the left gives increased lift, and the machine therefore heels over to the right. This is not sufficient for steady circling flight: the machine must have an angular velocity. This is produced by means of turning the rudder to the right. A steady circling motion presupposes a proper adjustment of the engine, elevator, ailerons and rudder.

The circular motion of a particle is completely defined if we know the radius of the circle and the speed. It is not so in the case of a rigid body. Firstly we must know the angular velocity of the body. In our case we take this to be the same as the angular velocity of the c.g. in its circular path, so that the whole of the motion can be represented by the machine rotating about the vertical through the centre of the circle. But we must know another thing in order to define the motion completely, and this is the position of the body

relative to the path of the c.g. Thus, if a straight rod is moving on a table so that its c.g. describes a circle, and the rod itself rotates about the centre of the circle with the same angular velocity, the position of the rod at any moment may be tangential to the circle, or it may cut the circle again in front of the c.g. or behind the c.g. If the rod is tangential to the circle, then at any moment the motion of the c.g. is along the rod itself; if, however, the rod is not tangential to the circle, then relative to the rod the c.g. moves sideways as well as forwards, so that, as seen from the rod, the c.g. has one component of motion along the rod itself, and another component, or *side-slip*, at right angles to the rod—towards the centre if the rod cuts the circle again behind the c.g., away from the centre if it cuts the circle again in front of the c.g.

The simplest as well as the most advantageous circling flight of an aeroplane is that in which there is no side-slip: as the pilot's seat is generally near the c.g. this means that the pilot feels the relative wind as coming from the front, and not sideways.

**15. Normal Circling Flight; Motion of C.G.**—We define normal circling flight as being that circular motion in which the propeller axis is horizontal, and there is no side-slip; thus the propeller axis touches the circular path of the c.g. We take the air-density to be that near the ground, and the loading to be that in the normal flight of § 2.

Fig. 7 shows the axes we shall use. The c.g. is the origin,  $G$ , and the  $X$ -axis is the forward direction of the propeller axis, as seen by the pilot; the  $Y$ -axis is the downward perpendicular to the  $X$ -axis in the longitudinal plane; the  $Z$ -axis is the line perpendicular to the  $X$ - $Y$ -plane, to the left of the pilot. These axes are fixed in the body of the machine, and move with it. The circular flight takes place in a circle in the horizontal plane containing  $GX$ , so that, as there is no side-slip, the centre of the circle is on the horizontal line through  $G$  perpendicular to  $GX$  and to the right as seen by the pilot. Continue this line beyond  $G$  and call this line  $Gz$ : it is obviously the position of  $GZ$  when there is no bank; draw  $Gy$  vertically downwards: this is the position of the  $Y$ -axis when there is no bank; it is clear that the  $Gx$ -axis for no bank coincides with  $GX$ . The bank is indicated by the angle  $\varphi$  between  $Gy$  and  $GY$ , measured positively in the sense  $Y \rightarrow Z$ .

We shall assume that the radius of the circular path is sufficiently large compared to the dimensions of the machine to warrant our neglecting the air-resistance effects due to the rotation. Also, in discussing the motion of the c.g. we shall omit the comparatively small effects of the elevator and rudder and of the displaced ailerons: in any case we have not yet information to give their positions. As there is no side-slip we can take the wing-pressure and the body-resistance to be the same as in the rectilinear flight. Let  $U$  be the velocity. The forces acting on the machine are easily seen to be

$W$  along  $\vec{Gy}$ ,

the propeller thrust  $T$  along  $\vec{Gx}$ ,

the body-resistance  $\left(\frac{U}{U_0}\right)^2 R_2$  along  $\vec{xG}$ , i.e.  $Gx$  backwards,

and the wing-pressure  $\left(\frac{U}{U_0}\right)^2 R_1$  in the longitudinal plane, at angle  $\alpha$  with  $\vec{YG}$ , i.e. with  $GY$  upwards.

The last force has a component  $\left(\frac{U}{U_0}\right)^2 R_1 \sin \alpha$  reinforcing the body-resistance, and a component  $\left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha$  along  $\vec{YG}$ ; and this latter gives a lift  $\left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha \cos \varphi$  to balance  $W$ , and a centripetal force  $\left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha \sin \varphi$

towards the centre of the path. If  $r$  is the radius of the path, then all the forces acting must be equivalent to a force  $WU^2/gr$  towards the centre of the path. Hence

$$\left. \begin{aligned} W &= \left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha \cos \varphi, & T &= \left(\frac{U}{U_0}\right)^2 R_1 \sin \alpha + \left(\frac{U}{U_0}\right)^2 R_2, \\ \frac{WU^2}{gr} &= \left(\frac{U}{U_0}\right)^2 R_1 \cos \alpha \sin \varphi. \end{aligned} \right\} \quad (21)$$

It is at once obvious that the necessary angle of bank is defined by

$$\tan \varphi = \frac{U^2}{gr}, \quad (22)$$

in which  $U, g, r$  are measured in the same length unit and  $U, g$  in the same time unit. Since  $W = R_1 \cos \alpha$ , we deduce  $U/U_0 = \sqrt{\sec \varphi}$ . Also, after a little algebra,

$$\left. \begin{aligned} H &= H_0 \sec^{\frac{3}{2}} \varphi; \\ \sin \varphi &= \frac{U_0^2}{gr}. \end{aligned} \right\} \quad (23)$$

Further, we find If then the pilot wishes to fly in normal circular flight with radius  $r$ , he must produce the angle of bank given by (23) and must use the power given in (23): the velocity will be  $U_0 \sqrt{\sec \varphi}$ .

As a numerical example take the machine of § 3. To get circular flight with  $r = 1$  furlong, we must have  $\sin \varphi = 11/30$ , so that  $\varphi = 21\frac{1}{2}^\circ$ ; the necessary power is  $52.8/(0.9304)^{\frac{3}{2}} = 58.7$ , and the duration of a circuit is about  $45\frac{1}{2}$  secs.

Since  $\sin \varphi$  must be less than unity, we have an *a priori* limit to the radius of the path, namely  $r > U_0^2/g$ . Of course this minimum cannot be reached, because the necessary power rises quickly. If  $H^p$  is the greatest power available, the minimum radius is given by

$$\frac{H^p}{H_0} = \left(1 - \frac{U_0^4}{g^2 r^2}\right)^{-\frac{3}{2}}, \quad i.e. \quad r = \frac{U_0^2}{g} \left[1 - \left(\frac{H_0}{H^p}\right)^{\frac{2}{3}}\right]^{-\frac{1}{2}}. \quad (24)$$

In our case the minimum radius is 320 ft.; the angle of bank is then just over  $49^\circ$ , the speed is 74 mls./hr., and a circuit is made in  $18\frac{1}{2}$  secs.

Smaller radii can be obtained with a more powerful engine. We shall examine later the effect of different angles of attack.

**16. Rotation about the C.G.**—It does not follow that because the conditions for the circular motion of the c.g. are satisfied, then the machine will fly steadily with circling flight. Some effort is necessary in order to maintain this state.

The machine is to have an angular velocity  $\Omega = U/r$  radians per second, where  $U$  and  $r$  are measured in the same unit of length. Consider an element of weight  $w$  at the point  $(x, y, z)$ , Fig. 7. Since this is to rotate about the  $y$ -axis with angular velocity  $\Omega$ , there must be a centripetal force  $\Omega^2 w \sqrt{x^2 + z^2}/g$  horizontally towards the  $y$ -axis; hence the moments of the external forces acting on the machine must be equivalent to the moments of the whole set of such forces for all elements of the aeroplane. The centripetal force for the element considered has components

$$\frac{\Omega^2 wz}{g} \text{ parallel to } \vec{zG} \quad \text{and} \quad \frac{\Omega^2 wx}{g} \text{ parallel to } \vec{xG},$$

giving about the axes  $x, y, z$  respectively the moments

$$-\frac{\Omega^2 w y z}{g}, \quad 0, \quad \frac{\Omega^2 w x y}{g}$$

reckoned positively in the senses  $y \rightarrow z$ ,  $z \rightarrow x$ ,  $x \rightarrow y$ . But we have the transformations

$$x = X, \quad y = Y \cos \varphi - Z \sin \varphi, \quad z = Y \sin \varphi + Z \cos \varphi.$$

where  $(X, Y, Z)$  are the coordinates of the point  $(x, y, z)$  referred to the axes  $GX, GY, GZ$ . The moments are therefore to be

$$-\frac{\Omega^2 w}{g}[(Y^2 - Z^2) \sin \varphi \cos \varphi + YZ \cos 2\varphi], \quad 0, \quad \frac{\Omega^2 w}{g}[XY \cos \varphi - XZ \sin \varphi].$$

Let  $A, B, C, D, E, F$  be the moments and products of inertia about the axes  $X, Y, Z$ , so that

$$A = \sum w(Y^2 + Z^2), \quad D = \sum wYZ, \text{ etc.}$$

The moments when summed for all parts of the aeroplane become

$$\frac{\Omega^2}{g}(B-C)\sin\varphi\cos\varphi - \frac{\Omega^2}{g}D\cos 2\varphi, \quad 0, \quad \frac{\Omega^2}{g}F\cos\varphi - \frac{\Omega^2}{g}E\sin\varphi.$$

Since  $X$ ,  $Y$  are in the longitudinal plane, which is a plane of symmetry except for the displacements of the rudder and ailerons, it is not very wrong to ignore  $D$  and  $E$  as compared with  $(B - C)$  and  $F$ . We hence require couples

$$\frac{\Omega^2}{g}(B-C)\sin\varphi\cos\varphi, \quad 0, \quad \frac{\Omega^2}{g}F\cos\varphi$$

about the axes  $x, y, z$ . About the axes  $X, Y, Z$  we need the couples

$$\frac{\Omega^2}{g} (B - C) \sin \varphi \cos \varphi, \quad \frac{\Omega^2}{g} F \sin \varphi \cos \varphi, \quad \frac{\Omega^2}{g} F \cos^2 \varphi. \dots \quad (25)$$

These couples are obtained by means of the *controls*: the flaps must be adjusted so that, including the couple due to the excess of pressure on the outer wing over that on the inner wing, they give the requisite *X* couple; the rudder to give the *Y* couple; the elevator to give the *Z* couple. In an actual machine *F* is positive; hence the elevator must be turned down; the rudder must be turned to the right for a right-hand turn.

17. **General Circling Flight.**—If we use different angles of attack we shall, in order to simplify the mathematics, suppose that the bank is in the form of a roll about the line through the c.g. along which the machine is moving: as there is no side-slip this means about the horizontal tangent to the circular path.

Let the propeller axis be at angle  $\beta$  with the horizontal tangent, so that the angle of attack is  $\alpha + \beta$ . If  $U$  is the speed we have a wing-pressure  $\left(\frac{U}{U_0}\right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1$  and a body-resistance  $\left(\frac{U}{U_0}\right)^2 R_2$ . Decomposing in the longitudinal plane horizontally and perpendicularly to the direction of flight, we have components

$$T \cos \beta - \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin(\alpha + \beta) - \left( \frac{U}{U_0} \right)^2 R_2 \cos \beta,$$

$$T \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) - \left( \frac{U}{U_0} \right)^2 R_2 \sin \beta.$$

If we now decompose the second component horizontally and vertically, we get the conditions:

$$W = \left[ T \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) - \left( \frac{U}{U_0} \right)^2 R_2 \sin \beta \right] \cos \varphi,$$

$$T \cos \beta = \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin(\alpha + \beta) + \left( \frac{U}{U_0} \right)^2 R_2 \cos \beta,$$

$$\frac{WU^2}{gr} = \left[ T \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) - \left( \frac{U}{U_0} \right)^2 R_2 \sin \beta \right] \sin \phi.$$

We easily deduce  $\frac{U^2}{gr} = \tan \varphi$ ,  $\left(\frac{U}{U_0}\right)^2 = \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta) \cos \varphi}$ .

If we approximate we get

$$\frac{U}{U_0} = \sqrt{\frac{\alpha + \beta}{\alpha + \beta} \sec \varphi}, \quad H = H_1 \sec^2 \varphi \cdot \left(\frac{U_0}{U}\right) + H_2 \left(\frac{U}{U_0}\right)^3; \quad \left. \right\} \quad (26)$$

also  $\tan \varphi = U^2/gr$ ;

$$\text{hence } H = H_1 \left(1 + \frac{U^4}{g^2 r^2}\right) \left(\frac{U_0}{U}\right) + H_2 \left(\frac{U}{U_0}\right)^3$$

$$\text{i.e. } H = H_1 \left(\frac{U_0}{U}\right) + H_1 \frac{U_0^4}{g^2 r^2} \left(\frac{U}{U_0}\right)^3 + H_2 \left(\frac{U}{U_0}\right)^3. \quad (27)$$

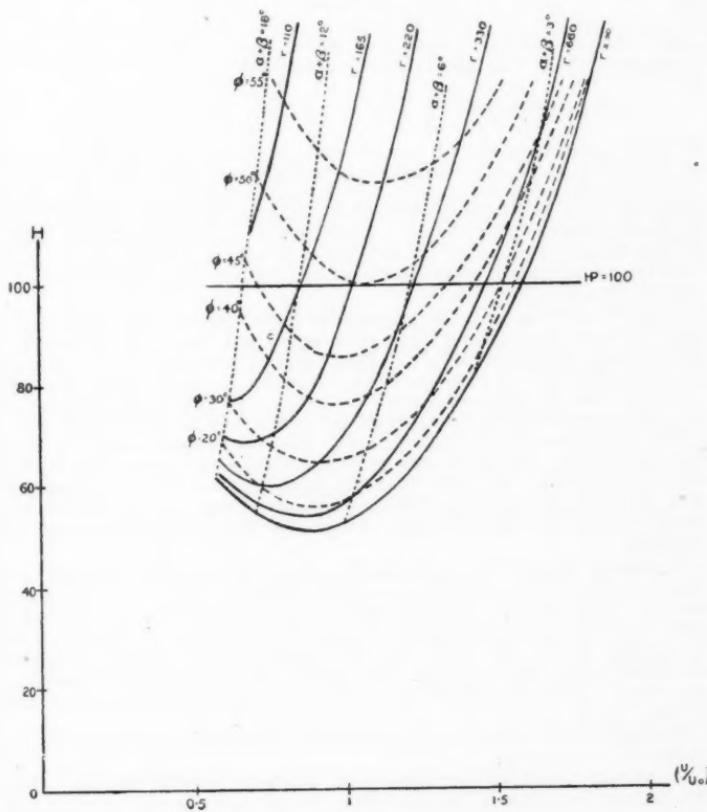


FIG. 8.

It is again useful to have recourse to graphical representation. In Fig. 8 we have plotted  $H$  against  $U/U_0$  for  $r = \infty$  (rectilinear flight, this being the same curve as in Fig. 3 (b)),  $r = 660, 330, 220, 165, 110$  ft. respectively. The

dashed curves give the angles of bank: these are plotted either from (26), which gives  $H$  in terms of  $\varphi$  and  $U/U_0$ , or by means of  $U^2 = gr \tan \varphi$ , so that

$$\left(\frac{U}{U_0}\right)^2 = \frac{gr \tan \varphi}{U_0^2}.$$

The dotted curves give the angles of attack: these are plotted by means of the equations:

$$\left(\frac{U}{U_0}\right)^2 = \frac{\alpha}{\alpha + \beta} \sec \varphi, \quad \tan \varphi = \frac{U^2}{gr},$$

so that

$$\left(\frac{U}{U_0}\right)^4 = \left(\frac{\alpha}{\alpha + \beta}\right)^2 \left[1 + \frac{U_0^4}{g^2 r^2} \left(\frac{U}{U_0}\right)^4\right],$$

and

$$\left(\frac{U}{U_0}\right)^4 = \frac{1}{\left(\frac{\alpha + \beta}{\alpha}\right)^2 - \frac{U_0^4}{g^2 r^2}}.$$

Another way is to use (26), whence we get

$$H = \left[ \left(\frac{\alpha + \beta}{\alpha}\right)^2 H_1 + H_2 \right] \left(\frac{U}{U_0}\right)^3,$$

the same curves as in Fig. 5. It is in fact easy to see that the  $\varphi$  curves are the same set as the curves in that figure. The physical justification is obvious.

We now have full information about the possibilities of circling flight. Using  $H^p = 100$ , we at once see that for normal circular flight, i.e.  $\alpha + \beta = 6^\circ$ , the smallest radius of the circuit is about 320 ft., with angle of bank nearly  $49^\circ$  and velocity 74 mls./hr. If we are not to have a larger angle of attack than  $18^\circ$ , the smallest radius possible is about 120 ft., with angle of bank  $43^\circ$ , velocity 40 mls./hr.

There is a well-known rule in practical flight, namely, that in changing from rectilinear to circling flight it is necessary to raise the nose of the machine. This is made obvious by Fig. 8. If the same power, say 80 H.P., is used in the circling flight, say with  $r = 330$  ft., as in the rectilinear flight, the angle of attack must be increased from  $3^\circ$  to about  $7^\circ$ . This is true in general: *the angle of attack must be increased if  $r$  diminishes while  $H$  remains fixed.*

The figure gives further information. Thus the greatest angle of bank that can be used with the machine of § 3 is  $\varphi = 50^\circ$ , with  $r =$  about 230 ft., and velocity 64 mls./hr. This and other such results can also be deduced analytically by using the equations of this paragraph. There are, in addition, the conditions that the controls shall be such that the necessary moments are obtained.

**18. Helical Flight; Zero Side-Slip.**—Just as climbing and descending are generalisations of horizontal flight, so helical climb and descent are generalisations of circling flight. In helical flight the c.g. of the machine describes a helix traced on a vertical circular cylinder. If the radius of the cylinder  $r$ , the velocity  $U$ , and the angle of ascent  $\theta$  (negative for descent) are given, the helical motion is fully defined. The motion of the machine can be described as a rotation about the axis of the cylinder with an ascent or descent parallel to this axis.

We shall consider briefly the conditions for such flight as far as the motion of the c.g. is concerned.

In order to make the mathematics as simple as possible we shall define the bank in the following manner. Let the c.g. move at any moment in a direction making an angle  $\theta$  with the horizontal. As there is no side-slip the line representing the velocity at any moment touches the cylinder of the helix at the c.g. Let the body of the machine turn round this line through an angle  $\varphi$ ; we call this the bank. The longitudinal plane contains this line: hence if the propeller axis is at an angle  $\beta$  with this line, the angle

of attack is  $\alpha + \beta$ . The forces acting on the machine are the weight vertically downwards,  $\left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\}$  along the forward direction of the propeller axis, and  $\left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1$  in the longitudinal plane perpendicular to the direction of zero wing-pressure. Take the components of the latter two forces along the momentary direction of flight of the c.g., and along the perpendicular to this in the longitudinal plane; we get respectively

$$\left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \cos \beta - \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin(\alpha + \beta),$$

and  $\left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta).$

Decompose the second component horizontally, which means towards the axis of the cylinder, and in the vertical plane containing the direction of motion of the c.g.; we get respectively

$$\left[ \left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) \right] \sin \varphi,$$

and  $\left[ \left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) \right] \cos \varphi.$

The latter component and the force along the direction of motion of the c.g. must balance the weight of the machine, whilst the component towards the axis of the cylinder is the centripetal force. Hence we have the following conditions :

$$W \cos \theta = \left[ \left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) \right] \cos \varphi,$$

$$\frac{WU^2}{gr} = \left[ \left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \sin \beta + \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos(\alpha + \beta) \right] \sin \varphi,$$

and  $W \sin \theta = \left\{ T - \left( \frac{U}{U_0} \right)^2 R_2 \right\} \cos \beta - \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \sin(\alpha + \beta).$

We get  $\frac{U^2}{gr} = \cos \theta \tan \varphi;$

also  $W(\cos \theta \cos \beta \sec \varphi - \sin \theta \sin \beta) = \left( \frac{U}{U_0} \right)^2 \frac{\sin(\alpha + \beta)}{\sin \alpha} R_1 \cos \alpha,$

so that  $\left( \frac{U}{U_0} \right)^2 = \frac{\sin \alpha}{\sin(\alpha + \beta)} (\cos \theta \cos \beta \sec \varphi - \sin \theta \sin \beta);$

further,

$$T = W \sin \theta \sec \beta + W \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} (\cos \theta \cos \beta \sec \varphi - \sin \theta \sin \beta) + \left( \frac{U}{U_0} \right)^2 R_2, \dots \quad (28)$$

To get  $H$  we multiply  $T$  by  $U \cos \beta / 375$ : we find

$$H = \frac{WU \sin \theta}{375} + H_1 (\cos \theta \cos \beta \sec \varphi - \sin \theta \sin \beta) \left( \frac{U_0}{U} \right) + H_2 \cos \beta \left( \frac{U}{U_0} \right)^3.$$

If now we approximate by making  $\alpha, \beta, \theta$  small, we get

$$\frac{U^2}{gr} = \tan \varphi, \quad \frac{U}{U_0} = \sqrt{\frac{\alpha}{\alpha + \beta} \sec \varphi}, \quad H = \frac{WU \theta}{375} + H_1 \sec^2 \varphi \left( \frac{U_0}{U} \right) + H_2 \left( \frac{U}{U_0} \right)^3. \quad (29)$$

The equation for  $H$  can be written in the same form as (27) for circling flight. Comparing (29) with (26) we see that on an exact analogy with the results in § 5 and § 7, the rate of vertical climb in helical flight is determined by

the reserve of power used over and above that required for circling flight with the same angle of attack and radius of circuit. If the reserve of power is negative we get the rate of vertical descent.

It follows that the rate of climb with any given power for prescribed radius is determined from Fig. 8 by taking the ordinate intercepted between the horizontal line representing the available power, and the  $H$  curve for the given radius. Thus for  $H' = 100$  and  $r = 330$  ft., we get for  $U = 60$  mls./hr. a rate of climb 480 ft./min. The maximum rate of climb with this radius is 660 ft./min., as can also be verified by calculation: the velocity is then 45 mls./hr. and the angle of climb  $91^\circ$ .

If we write (29) in the form

$$H = H_1 \frac{\theta}{\alpha} \left( \frac{U}{U_0} \right) + H_1 \sec^2 \varphi \left( \frac{U_0}{U} \right) + H_2 \left( \frac{U}{U_0} \right)^3,$$

we gather that the method of § 9 can be applied to helical flight, with the curves in Fig. 8.

It is assumed, of course, that the controls are in such a configuration that the couples necessary for the motion are obtained.

It is worth noticing that the curves in Fig. 8 give the complete information about the performance of the aeroplane for which it is plotted. Thus the curve  $r=\infty$  takes the place of the curve in Fig. 3 (b). The  $\phi$  curves are really the  $W'/W$  curves in Fig. 5, using the relation  $W'/W = \sec \phi$ . It follows that the information given by Fig. 2 (b) is also contained in Fig. 8. Further, since we have seen that the  $\rho'/\rho$  curves in Fig. 6 are similar about the origin, we can replace them all by the single one  $\rho'/\rho = 1$ , i.e. the  $r=\infty$  curve in Fig. 8, using for the available power at any height for which the density is  $\rho'$  the expression  $H^p(\rho'/\rho)^{\frac{3}{2}}$ . This is because the different  $\rho'/\rho$  curves (15) can be written in the form

$$H/\left(\frac{\rho}{\rho'}\right)^{\frac{1}{2}} = H_1 \frac{\left(\rho/\rho'\right)^{\frac{1}{2}}}{(U/U_0)} + H_2 \frac{(U/U_0)^3}{\left(\rho/\rho'\right)^{\frac{3}{2}}},$$

so that their dimensions are proportional to  $(\rho/\rho')^{\frac{1}{2}}$ . Hence, if we replace them by the curve  $\rho' = \rho$ , we change their dimensions in the ratio  $(\rho'/\rho)^{\frac{1}{2}}$ . If we multiply this into  $H\rho'/\rho$  we get the expression suggested. This is a very simple way of dealing with the ceiling.

An extraordinary amount of information is thus afforded by the set of graphs in Fig. 8.

19. **Helical Glide.**—Put  $H = 0$  in (29), and we get a value of  $\theta$  which represents the gliding angle for a helical glide. For small values of  $\theta$  we easily get that

$$\left( \alpha + \beta + \frac{\alpha^2}{\alpha + \beta} \frac{H_2}{H_1} \right) \sec \varphi \quad \dots \dots \dots \quad (30)$$

is the gliding angle for angle of attack  $\alpha + \beta$  and angle of bank  $\varphi$ , it being assumed that the necessary couples are produced by the flaps, rudder and elevator. If  $\sec \varphi$  is expressed in terms of  $(\alpha + \beta)/\alpha$  and  $r$  by means of (29), we get the gliding angle in terms of the radius of the circuit and the angle of attack. If we compare (30) with (8) we see that a helical glide is always steeper than the rectilinear glide with the same angle of attack.

The vertical fall in a circuit, in other words the pitch of the helix, is easily found to be

$$\frac{2\pi\alpha U_0^2}{g \sin \varphi \cos \varphi} \left[ 1 + \left( \frac{\alpha}{\alpha + \beta} \right)^2 \frac{H_2}{H_1} \right],$$

where  $\alpha$  is in radians.

For the machine of § 3 this becomes

$$\frac{320}{\sin 2\varphi} \left[ 1 + \frac{4}{7} \left( \frac{\alpha}{\alpha + \beta} \right)^2 \right].$$

Thus with neutral elevator and  $\varphi = 45^\circ$ , the pitch is about 500 ft., the radius being 340 ft.

Equation (30) only holds for moderate gliding angles: if the glide is steep we get a more accurate result by going back to (28), putting  $T = 0$ . There is no difficulty about finding the angle of glide and the corresponding radius.

20. The effects of variable loading and different air-densities can be examined in the manner already described in § 11 and § 12, whilst circular and helical flight in a wind present many geometrical problems of great interest. The author hopes to develop the subject in greater detail elsewhere: the present paper will perhaps suffice as an indication to teachers of mechanics and others, of how much really useful and interesting information can be obtained with easy mathematics. The introduction of aeroplane mathematics into ordinary courses at our schools and universities would be a great boon to teachers as well as to pupils.

S. BRODETSKY.

### GLEANINGS FAR AND NEAR.

74. Johnson read a good deal of . . . Gregory, and I observed he made some geometrical notes in the end of his pocket book.—Boswell's *Hebrides*, Oct. 7, 1773.

[At Island of Coll. Possibly David Gregory's *Treatise of Practical Geometry*, translated with additions by Colin Maclaurin, of which a 7th ed. appeared in 1769.]

75.

Well the old weary year hath flown,  
With all its wars and horrid panic . . .  
And wars are over in a week,  
Cost half a thousand crowns a minute!  
While Palliser lays ironclads low,  
As does De Morgan circle-squarers . . .  
Here's sixty seven who comes to vow  
We're all at sixes and at sevens. . . .

*N. & Q. IV. ii. 28 (Verses on '66 and '67).*

76. No lady reads a novel with more anxious intent than a mathematical investigator a problem, particularly if from some new and untilled field of research. All the energies of his mind are called forth, all his faculties are on the stretch for the discovery.—Graves' *Life of Hamilton*, i. 114.

77.

I next with rapture view'd the meadow round  
Which I—an oblique plain triangle found.

—John Lewis, Schoolmaster of Syston, *London Mag.* 1750.

78. **Kingsley Letters Home.**

I am getting on in algebra. I am studying mathematics, and Mr. Coleridge says he is pleased at my liking it so much. [Derwent, son of S. T. Coleridge the poet.]

*May 31, 1839 (Magdalene).*

You will be delighted to hear that I am *first* in classics and mathematics also, at the examinations, which has not happened in the college for several years . . . Mr. Wand [? Waud] has offered to help me with my 2nd year's subjects, so I shall read conic sections and the spherical trigonometry very hard while I am here.

*Helston, February, 1836.*

*Cam., Feb. 13, 1842.*

I came out to my great astonishment, and that of my tutor, a tolerable second-class, with my little reading. [He got a first in classics.]

## REVIEWS.

**A. S. Gomes dê Carvalho. A Teoria das Tangentes antes da Invenção do Cálculo Diferencial.** 1919. (Coimbra : imprensa da Universidade.)

An interesting dissertation in three chapters. I. and II. deal respectively with the treatment of tangency in the golden period of Greek Geometry and in the middle of the seventeenth century A.D. Fully worked examples are given, the author, like Prof. Heath, using modern notation where the prolixity of the ancient investigations might deter readers with little time to spare. These examples seem well chosen, those in particular of the work of Apollonius illustrating at once the originality and beauty of his results and the tedious methods by which he produces them. A sketch is added of the famous treatment of the "four normals from a point" and the determination of points on the evolute of a conic. Among the moderns Fermat, Descartes, Roberval, and Barrow have the most space allotted to them, the disputes between the first two and between Roberval and Torricelli being discussed. Hudde, Huygens, De Sluze, Wallis, Tschirnhausen, Fatio de Duillier, De l'Hôpital have honourable mention. Chapter III. treats briefly of the inverse problem of tangents, that is the investigation of curves arising from known properties of their tangents in some of the few cases solved before the invention of the Integral Calculus reduced it to the solution of differential equations.

E. M. LANGLEY.

**Physics. The Elements.** By N. R. CAMPBELL, Sc.D. Pp. x + 566. Price 40s. net. 1920. (Cambridge University Press.)

As a fair sample of the quality of this work we may take the chapter on probability. The author properly points out what mathematicians have not always remembered ; namely, that everything ultimately depends upon the assumption of "equally probable" cases : he also makes some very sensible remarks about such terms as "random," "independent," and so on. But in discussing a particular example on p. 166 he comes sadly to grief, and it is difficult not to accuse him of quibbling. A piquet pack and a whist pack are placed side by side on a table, and a card is drawn at random ; what is the chance of drawing an ace of spades ? The orthodox answer is  $21/832$  ; Dr. Campbell does not dispute the correctness of this, but asks what are the 832 equally favourable cases corresponding to this fraction—the possible number of events being 84, the total number of cards. It seems as if the author required *every* probability  $p/q$  to be explainable in relation to some  $q$  equally probable cases ; thus ignoring, or cavilling at, the whole theory of compound events. An extreme case may be given. A person is allowed to choose either of two purses, and draw one coin from it. One purse contains  $m$  shillings, and the other  $n$  sovereigns ; what is the chance of drawing a sovereign ? Clearly  $\frac{1}{2}$  ; but the number of events is  $(m+n)$ . There is no contradiction, because the events are not equally probable : the probability of drawing a particular shilling is  $1/2m$ , and that of drawing a particular sovereign is  $1/2n$ .

Another characteristic chapter is that on units and dimensions. It is a strange mixture of pertinent criticisms and diffuse rambling arguments which are almost, if not quite, paradoxical. The criticism (p. 406) of the theory of dimensions applied to the formula for the period of a simple pendulum is perverse and irrelevant ; it would be waste of space to justify this statement here.

It is distressing to be obliged to make these unfavourable criticisms. Dr. Campbell has been impressed by, and to some extent has comprehended, the critical work of Peano and his school in the domain of pure mathematics ; with the idea of performing a similar task for physics, he has spent a vast amount of labour, too honest and independent to be wholly wasted, but often misapplied and ineffective. Applied mathematics may ultimately, perhaps, be reduced to a purely logical system, but the prospect, at present, seems very remote. Even in geometry there are things like the connectivity of surfaces, which are only with great difficulty made amenable to analytical treatment ; and even when this is successful, most people, I fancy, will feel

dissatisfied with the highly artificial character of the result. In physics the difficulty of arithmetising the theory is much greater, because we have so many "qualities" or "properties" to deal with, and even assuming that we can measure each of them separately, there is still the question of their mutual relations. For instance, in the mathematical theory of electricity, quantities of vitreous and resinous electricity are merely distinguished by the signs + and -, just like readings on a thermometer; but there can hardly be any doubt that there is an intrinsic difference between the two sorts of electrification more complex than can be completely expressed by a variation of sign.

A competent physicist will probably read Dr. Campbell's book with the same sort of combination of interest and amusement, and occasional exasperation, as a competent philosopher (like Mr. F. H. Bradley, for instance) feels when reading Herbert Spencer's *First Principles*. It is doubtful whether an untrained student would derive any benefit from it, and it might do him harm by bewildering him.

Finally, it should be noted that Dr. Campbell has given new meanings of his own to certain current technical terms. Of course, he has a perfect right to do this; but it is rather unfortunate that one of these terms is "concept." English is very poor in philosophical terms, and it is a pity to use "concept" in a new sense, especially in a work where the word, in its usual sense, would be so convenient.

G. B. MATHEWS.

**A Course of Modern Analysis.** Third edition. By E. T. WHITTAKER and G. N. WATSON. Pp. 608. 40s. net. 1920. (Camb. Univ. Press.)

Encyclopaedic in scope, this treatise is one of the notable achievements of Cambridge scholarship in our time, and the title is as apt now as it was in 1902. But in this edition only two chapters call for comment, for elsewhere the work is essentially a reprint of the edition of 1915, and the authors have not attempted to meet criticisms: for example, not only is the clumsy theory of real numbers in the first chapter repeated, but it is still described as Russell's.

The chapter on trigonometrical series has been rearranged and revised. By the revision, in which in particular the proof of Fejér's fundamental theorem has been simplified, we can all benefit. The rearrangement is for the sake, we are told, of students interested primarily in applications, but it is possible to doubt whether physicists and applied mathematicians will turn even now to these pages to make the acquaintance of Fourier series, for the fascination that made the first edition of *Modern Analysis* a delight to schoolboys and a purifier of the mechanically-minded has evaporated.

The other substantial change is the addition of an irritating and inspiring chapter on ellipsoidal harmonics and Lamé's equation. The elementary formulae relating to confocal coordinates might surely have been quoted, since the variables are restricted quite unnecessarily to be real; nothing either novel or elegant is said on this head. In both manner and matter the analysis leading to the construction of ellipsoidal harmonics is childish in the extreme: it is simpler and far more instructive to discover the condition for the function  $x^l y^m z^n \Lambda(\lambda) \Lambda(\mu) \Lambda(\nu)$  to satisfy Laplace's equation without imposing conditions on the function  $\Lambda(\lambda)$  or restricting the values of the indices than to rely on the hypothesis that  $\Lambda(\lambda)$  is a polynomial without repeated factors and to deal severally with certain special sets of values of  $l, m, n$ , especially as it is the former course that reveals which sets of values of  $l, m, n$  enable the latter to succeed. The second half of this chapter shows the authors at their best, bringing to bear on the solution of Lamé's equation a great variety of methods, and making a skilful combination of results due to a number of independent workers. And the student to whom earlier chapters have conveyed the impression that nothing in analysis remains to be done will learn at last of one equation in which a parameter has not yet been generalised! E. H. N.

**Géométrie et Analyse des Intégrales doubles.** By A. BUHL. Pp. 67. 1919. (Scientia, No. 36; Gauthier-Villars.)

This booklet is concerned with surface integrals transformable into contour integrals by formulae such as those of Riemann and Stokes. Applications of the transformations are made to differential geometry, to Abelian questions,

and to the theory of partial differential equations of the Monge-Ampère type, and there is persistent reference to the author's own memoirs. The value of the methods is beyond doubt, but they would be recommended better by a more discriminating advocate : for example, the proof of the Gauss-Bonnet relation between geodesic curvature and total curvature is as ugly as is to be expected if the surface is taken as  $z=f(x, y)$  and the contour as a section of the cylinder  $F(x, y)=0$ .

No reason is given for reversing the ordinary convention that principal curvatures of a surface are measured in the positive direction of the normal. In the absurdity of using the name of *mean* curvature for the sum of the principal curvatures and not for the half of this sum, Prof. Buhl is in eminent company, but he can claim neither the support of Darboux nor the authority of Sophie Germain, whose argument when first she drew attention to the function was that if  $m$  is not less than two, the arithmetic mean of the normal curvatures along  $m$  tangents radiating at equal angular intervals of  $\pi/m$  is unaffected by a rotation of the group of tangents round the normal, and is independent of the value of  $m$ .

If the treatment of the subject shows no sense of proportion, the crudity of the language is shocking even to a foreigner. To say that the work does not deserve a place in a series with the reputation of " *Scientia* " is mild criticism.

E. H. N.

(1) **Principes Usuels de Nomographie avec applications à divers problèmes concernant l'Artillerie et l'Aviation.** By Lieut.-Col. D'OCAGNE. Pp. 67. Price 4 f. 50 c. *maj. temp.* 100 %. 1920. (Gauthier-Villars.)

(2) **A First Course in Nomography.** By S. BRODETSKY. Pp. xii + 135. 10s. net. 1920. (G. Bell.)

(1) From the practical point of view *Nomography* may be described shortly as that branch of applied mathematics which is concerned with the graphic representation of formulae.

M. d'Ocagne invented the French word and first used it in his *Nomographie*. *Les Calculs usuels effectués au moyen des Abaques*, which appeared in 1891. Since then he has published a number of works on the subject, notably his *Traité de Nomographie* (1899) and *Calcul Graphique et Nomographie* (1908). In his latest contribution, the title of which forms the heading of this note, he gives us the following definitions :

" La *nomographie* est la théorie générale de la représentation graphique coté des équations à un nombre quelconque de variables ; le but d'une telle représentation est de remplacer toute espèce de calcul numérique par de simples lectures faites sur des graphiques.

Tout graphique coté ainsi disposé est dit un *abaque* ou *nomogramme*."

Here we have the heart of the matter condensed into a few lines. We notice that the graphic representation comprises *figured scales*, and that the answer is read directly off it.

It is also advisable to note a distinction which the author makes clear in the Introduction to his *Traité de Nomographie*. To quote his words—

" Réduire à des simples lectures sur des tableaux graphiques, construites une fois pour toutes, les calculs qui interviennent nécessairement dans la pratique des divers arts techniques, tel est le but que se propose la *Nomographie*." And he adds the following note to " construire une fois pour toutes "— " Ce caractère, absolument fondamental pour la Nomographie, la distingue du Calcul graphique proprement dit, . . . Calcul auquel se rattache notamment la Statique graphique, et dans lequel, pour une opération effectuée sur des données particulières, on substitue à un calcul numérique le tracé d'une épure. On doit chaque fois pour un choix différent des données, recommence cette épure. Les abaques, au contraire fournissent, à la fois, le résultat d'une certaine opération pour tous les états possibles des données compris à l'intérieur d'un certain champ de variation."

The invention and introduction of some of the methods employed in Nomography date back to a remote period, but there can be no dispute as to the commanding position to be assigned to the work of M. d'Ocagne in the

generalisation and systemisation of the treatment. Other writers on the subject in various languages have drawn their inspiration from him and followed more or less closely in his footsteps.

The pamphlet now before us will be welcomed both for the masterly summary of the principles in this author's customary lucid style, and for the interest and elegance of the applications.

After a preliminary dissertation on scales he proceeds to the familiar graphic representation in cartesian coordinates of equations comprising two and three variables, the "abaque cartésien." He then passes to the less familiar representation in parallel coordinates, and so introduces us to the "nomogramme à points alignés," in which the result is read by the alignment of points on a straight line instead of by the intersection of lines in a point as in the "abaque cartésien."

The advantage of the "alignment diagram" over the "intersection diagram" is that any interpolation by eye has only to be made on simple graduated scales instead of between curves, and the whole diagram is much clearer, as it is not confused by various systems of lines or curves intersecting one another. Unfortunately, while any three-variable equation can be represented by an intersection diagram, only certain types can be represented by an alignment diagram, but they include a large number of those which are met with in practical formulae.

Further, M. d'Ocagne develops the combination of the two types in the "nomogramme à réseau de points à deux cotes," where for instance we may have a network of two systems of figured curves added to two rectilinear scales, the whole forming a direct reading four-variable nomogram, which would not be possible in a cartesian representation alone.

Throughout the pamphlet the principles are illustrated by a series of admirable examples of their application to formulae used in gunnery, ballistics, and aviation, but these examples, from their novelty and ingenuity, cannot fail to be of general interest in stimulating ideas for applications in other branches of applied science. In addition some valuable hints are given for the practical construction of nomograms.

To sum up, this is a fascinating little book, and can be freely recommended both to furnish fresh ideas to those already interested in the subject or, to those to whom it is novel, as an introduction with a view to arousing such an interest.

(2) Dr. Brodetsky, in his *First Course in Nomography*, deals only with M. d'Ocagne's "nomogrammes à points alignés." This restriction seems to the present writer to be a mistake, particularly in a work which purports to be an introduction to the subject. Without some preliminary notice, however short, of cartesian representations, the connection between the two types of representation through the Principle of Duality, which is so vital and so illuminating, drops out entirely.

Nomography is essentially a practical subject studied principally for its applications, in so far as they form working tools. For this reason it is considered that the examples from the start and throughout should be of a thoroughly practical nature. Some of Dr. Brodetsky's examples are very interesting, but the student is likely to be wearied and perhaps discouraged by the great length and detail of the explanations and illustrations of operations involving only simple addition, subtraction and multiplication, purposes for which it would generally be a misuse to employ nomograms in practical work. The student will have to grasp the idea of the "functional scale" to grapple with any really useful applications, and the sooner it is introduced the better. In contrast to Dr. Brodetsky, M. d'Ocagne makes it a starting point in all his works.

As a "First Course" the present writer would much prefer the *Principes Usuels*, the student going straight to the examples and not confusing or tiring himself too much at the start with the general theory. The interest excited by the examples and others that were suggested to him or which he could make for himself, in any line of investigation to which he was especially attracted, would then almost inevitably urge him back in due course to attack the general theory with renewed vigour and success.

R. K. HEZLET.

**The Theory of Determinants in the Historical Order of Developments.**  
By SIR THOMAS MUIR, C.M.G. Vol. III. The Period 1861-1880. Pp. xxvi + 503. 35s. 1920. (Macmillan.)

The author is to be warmly congratulated on having brought his monumental work to within a measurable distance of completion. We who sit at home at ease, within reach of great libraries, will find it difficult to realise how far distance from the means of reference handicaps a writer and lengthens the task involved in such a work as this. On his rare visits to England he has not always been able to discover, save by multitudinous enquiries or by a happy chance, where complete or partial sets of this or that continental periodical are to be found, although the catalogue published by the Mathematical Association must have made his task the easier. To have brought up the subject through another couple of decades, and that while preoccupied by the work of the high post he holds in South Africa, in the atmosphere created during a great war, and in spite of personal loss, shows a spirit of devotion and fortitude which we cannot but admire. We learn from the preface that the last volume, bringing the record up to the end of the nineteenth century, was nearly complete in manuscript two years ago.

Those who have not had the privilege of consulting the preceding volumes must be reminded that not the least valuable part of the author's labours is in the running critical analysis of every book and memoir recorded—and that these reveal an unequalled mastery of the subject and its history. The prominence given to the subject during the period is shown by the appearance of no less than sixty text-books. Among the earliest of these is the Treatise by C. L. Dodgson, succinctly characterised as "a text-book quite unlike all its predecessors," not because it retains any of the flavour of *Alice in Wonderland*, published a couple of years before, but because "professedly its main aim is logical exactitude." The year before the same versatile writer had published a very interesting condensation-process for the evaluation of determinants, whose elements are arithmetical. Sir Thomas Muir points out that applying Dodgson's rule to a determinant with general elements the process is better understood, and we obtain the necessary justification for its use. There is not a page in the book from which the student will not have occasion to bless his stars that Sir Thomas was not content to be a mere reporter.

**The Early Mathematical Manuscripts of Leibniz.** Translated from the Latin Texts published by Carl Immanuel Gerhardt, with Critical and Historical Notes. By J. M. CHILD. Pp. vi + 238. 7s. 6d. net. 1920. (Open Court Publishing Co.)

It was a happy thought of the late English editor of the *Monist* to pay a tribute to the memory of Leibniz, the greatest general scholar of his time, upon the occasion of the two hundredth anniversary of his death. The translation we have here before us appeared in sections from 1916 onwards, and are here gathered together in a convenient form with a full apparatus of notes, critical and historical, bibliography, etc. No one would envy Gerhardt the task he undertook when he published in the late forties the famous collection of holographs preserved with religious care in the Royal Library of Hanover. It has been well said that no man ever wrote with more care, no man ever blotted and altered and copied more than Leibniz. In the great collection, for instance, there were cases in which he had written a letter three times over, and finally amended it so much as to be obliged to give it to his secretary to make the last copy. One would imagine that the subject of so much thought must have been of the highest importance, but this sort of thing would occur in matters of little moment. Still, Leibniz was one of those men who could find time for everything, and this may have been one of the reasons why he surpassed his greater rival in the extent and variety of his acquirements.

The late Mr. P. E. B. Jourdain's choice of Mr. Child was justified. The interest of the papers here translated largely lies in the extent to which they satisfy the quest for the earliest date at which traces are to be found of the great ideas with which the name of the writer is associated, and for the gradual

appearance of signs of the influence upon Leibniz of the mathematical ideas of Isaac Barrow. Mr. Child came fresh to his study of these papers from an exhaustive examination of the mathematical works of Newton's great predecessor. He at once recognised the obligations of the younger student to the English master. To Barrow, Leibniz owed "everything but his methods." Leibniz took up Descartes, laid him down, just as Newton laid aside *The Elements*. He resumed his study of the great Frenchman, and applied the Cartesian geometry to the theorems of Barrow; meanwhile his own notation began to develop. The operational calculus began to take form. The critical and historical notes in this volume display complete familiarity with the great controversy which raged around the notorious *Commercium Epistolicum*. Finally Mr. Child is forced to the conclusion that "Leibniz was in no way indebted to Newton for anything," and that "he was under no obligation to Barrow for his methods." The translation runs smoothly, and a useful and necessary piece of historical work has been done. The prevailing characteristic of the commentary is one of intense enthusiasm, and at times one almost feels that the writer can scarcely refrain from regarding his own hypotheses as proven facts. But on the whole there are very few slips, and the only fault that we have to find with this interesting volume lies in those places wherein his excitement and anxiety to press home his point, the author indulges in colloquialisms which are unusual in works of this type.

**Oeuvres Complètes de Christiaan Huygens.** Publiées par la Société Hollandaise des Sciences. Vol. XIV. Calcul des Probabilités, Travaux de Mathématiques, 1655-1666. Pp. 556. N.p. 1920. (Nijhoff, La Haye.)

One third of this volume is given up to Huygens' work on Probabilities. The papers between 1655 and 1659 deal with problems and theorems of Arithmetic, Stereometry, and Analytical Geometry, work on the Theory of Numbers, including papers on the Pellian Equation; rectification of the parabola, quadratures of the curved surfaces of the three conoids; areas of curves, volumes of solids of revolution, centres of gravity; the cycloid; evolutes. Some of the mathematical results obtained by Huygens, and never published by himself, were added to the Commentaries of van Schooten on the *Geometria* of Descartes (1649 and 1659). These fill about sixteen pages. Finally the papers between 1661 and 1666 deal with logarithms and the logarithmic curve, quadrature of the hyperbola by logarithms with application to the height of the barometric column; the construction of the regular heptagon; constructions for tangents to algebraic curves, and for the diameter of a spherical surface; researches on cubics; calculation of the smallest number which divided by given numbers leaves given remainders. We shall be grateful to any reader who is familiar with the astronomical volumes of this collected edition for a reference to the passage where Huygens speaks of an afternoon observation as taken "après-diner."

**Elementary Algebra.** By C. V. DURELL and G. W. PALMER. Part I. Pp. xxxi + 256 + xlvi. With Introduction and complete set of Answers. 4s. 6d. Without Introduction, and with Answers to Questions where intermediate work is required (with perforated pages). 3s. 6d. 1920. (Bell & Sons.)

After using this little book for some months we have come to the conclusion that it is nearer the ideal book for beginners than any we have yet seen. Prominence is given to oral work, there is a reduction to a minimum of the usual "talk," the constant attention is paid throughout to the hundred and one minutiae which, if properly attended to at an early stage, lead to clearness of thinking, correctness, and even elegance of expression. All the familiar traps are here. Every master will recognise at once the good qualities of the book, and we know of nothing better to place in the hands of a private student. A boy who has worked it through will have a sound knowledge of the subject up to quadratics, and will be able to apply his powers to the problems that meet us in every-day life. There are good sets of revision papers, and, what is not often found in a book for beginners, a glossary.

**A Short Course in College Mathematics.** By R. E. MORITZ. Pp. ix + 236. 10s. 6d. net. 1919. (The Macmillan Company.)

This course consists of thirty-six lessons on Algebra, Co-ordinate Methods, and Plane Trigonometry. The value of such a general sketch of the more prominent features of subjects likely to be of use to the members of Army and Navy Students' Training Corps, and required by them in a great emergency, is obvious enough, and this book compares well with others compiled with the same object in view. It is not quite clear in what type of British schools such a textbook will prove just what is needed, but it may suffice to say that the portion of the text allotted to graphs is extremely well supplied with diagrams and lucidly set forth, while half the book is given to a very satisfactory exposition of as much Trigonometry as average boys are likely to require—this subject taking up half the chapters in the book.

**The Arithmetic of the Decimal System.** By J. CUSACK. Pp. xvi + 492 + 61. 1920. (Macmillan.)

The object of the author is to show, what a Committee of the House of Commons has recently doubted:—"That a very moderate change will provide us with a system of Money, Weights and Measures, and of calculation based on them, that will make the study of Arithmetic a pleasure instead of a drudgery to the child at school, and will render business calculations simple, intelligible, accurate and expeditious." He takes as standards the pound sterling, the pound weight, and the gallon, and builds up his system of calculations on these and their tenth, hundredth and thousandth parts. The only fractions in the book are decimal fractions. About a hundred pages are devoted to the one of the decimal systems of which we hear most in these days. The book is too voluminous for the school boy, but he is usually well versed in the art of skipping. The very large collection of examples with their answers will be acceptable to teachers and examiners.

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*Archimedes*, edited by D. Rivalta, Paris, 1615.

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Also Sir Thomas Muir for the gift of the third volume of his *Theory of Determinants*.

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